

A DICHOTOMY THEOREM FOR THE GENERAL MINIMUM COST HOMOMORPHISM PROBLEM

RUSTEM TAKHANOV¹

¹ Department of Computer and Information Science, Linköping University
E-mail address: g-rusta@ida.liu.se
E-mail address: takhanov@mail.ru

ABSTRACT. In the constraint satisfaction problem (*CSP*), the aim is to find an assignment of values to a set of variables subject to specified constraints. In the minimum cost homomorphism problem (*MinHom*), one is additionally given weights c_{va} for every variable v and value a , and the aim is to find an assignment f to the variables that minimizes $\sum_v c_{vf(v)}$. Let $MinHom(\Gamma)$ denote the *MinHom* problem parameterized by the set of predicates allowed for constraints. *MinHom*(Γ) is related to many well-studied combinatorial optimization problems, and concrete applications can be found in, for instance, defence logistics and machine learning. We show that *MinHom*(Γ) can be studied by using algebraic methods similar to those used for CSPs. With the aid of algebraic techniques, we classify the computational complexity of *MinHom*(Γ) for all choices of Γ . Our result settles a general dichotomy conjecture previously resolved only for certain classes of directed graphs, [Gutin, Hell, Rafiey, Yeo, European J. of Combinatorics, 2008].

1. Introduction

Constraint satisfaction problems (*CSP*) are a natural way of formalizing a large number of computational problems arising in combinatorial optimization, artificial intelligence, and database theory. This problem has the following two equivalent formulations: (1) to find an assignment of values to a given set of variables, subject to constraints on the values that can be assigned simultaneously to specified subsets of variables, and (2) to find a homomorphism between two finite relational structures A and B . Applications of *CSP*s arise in the propositional logic, database and graph theory, scheduling and many other areas. During the past 30 years, *CSP* and its subproblems has been intensively studied by computer scientists and mathematicians. Considerable attention has been given to the case where the constraints are restricted to a given finite set of relations Γ , called a constraint language [5, 8, 17, 25]. For example, when Γ is a constraint language over the boolean set $\{0, 1\}$ with four ternary predicates $x \vee y \vee z$, $\bar{x} \vee y \vee z$, $\bar{x} \vee \bar{y} \vee z$, $\bar{x} \vee \bar{y} \vee \bar{z}$ we obtain 3-SAT. This direction of research has been mainly concerned with the computational complexity of *CSP*(Γ) as a function of Γ . It has been shown that the complexity of *CSP*(Γ) is highly connected with relational clones of universal algebra [17]. For every constraint language Γ , it has been conjectured that *CSP*(Γ) is either in P or NP-complete [8].

1998 ACM Subject Classification: F.4.1, G.2.2, I.2.6.

Key words and phrases: minimum cost homomorphisms problem, relational clones, constraint satisfaction problem, perfect graphs, supervised learning.

In the minimum cost homomorphism problem (*MinHom*), we are given variables subject to constraints and, additionally, costs on variable/value pairs. Now, the task is not just to find any satisfying assignment to the variables, but one that minimizes the total cost.

Definition 1.1. Suppose we are given a finite domain set A and a finite constraint language $\Gamma \subseteq \bigcup_{k=1}^{\infty} 2^{A^k}$. Denote by $\text{MinHom}(\Gamma)$ the following minimization task:

Instance: A first-order formula $\Phi(x_1, \dots, x_n) = \bigwedge_{i=1}^N \rho_i(y_{i1}, \dots, y_{in_i})$, $\rho_i \in \Gamma$, $y_{ij} \in \{x_1, \dots, x_n\}$, and weights $w_{ia} \in \mathbb{N}$, $1 \leq i \leq n, a \in A$.

Solution: Assignment $f : \{x_1, \dots, x_n\} \rightarrow A$, that satisfies the formula Φ . If there is no such assignment, then indicate it.

Measure: $\sum_{i=1}^n w_{if(x_i)}$.

Remark 1.2. Note that when we require weights to be positive we do not lose generality, since $\text{MinHom}(\Gamma)$ with arbitrary weights can be polynomial-time reduced to $\text{MinHom}(\Gamma)$ with positive weights by the following trick: we can add s to all weights, where s is some integer. This trick only adds ns to the value of the optimized measure. Hence, we can make all weights negative, and $\text{MinHom}(\Gamma)$ modified this way is equivalent to maximization but with positive weights only. This remark explains why both names *MinHom* and *MaxHom* can be allowed, though we prefer *MinHom* due to historical reasons.

MinHom was introduced in [15] where it was motivated by a real-world problem in defence logistics. The question for which directed graphs H the problem $\text{MinHom}(\{H\})$ is polynomial-time solvable was considered in [12, 13, 14, 15, 16]. In this paper, we approach the problem in its most general form by algebraic methods and give a complete algebraic characterization of tractable constraint languages. From this characterization, we obtain a dichotomy for *MinHom*, i.e., if $\text{MinHom}(\Gamma)$ is not polynomial-time solvable, then it is NP-hard. Of course, this dichotomy implies the dichotomy for directed graphs.

In Section 2, we present some preliminaries together with results connecting the complexity of *MinHom* with conservative algebras. The main dichotomy theorem is stated in Section 3 and its proof is divided into several parts which can be found in Sections 4–8. The NP-hardness results are collected in Section 4 followed by the building blocks for the tractability result: existence of majority polymorphisms (Section 5) and connections with optimization in perfect graphs (Section 6). Section 7 introduces the concept of *arithmetical deadlocks* which lay the foundation for the final proof in Section 8. In Section 9 we reformulate our main result in terms of relational clones. Finally, in Section 10 we explain the relation of our results to previous research and present directions for future research.

2. Algebraic structure of tractable constraint languages

Recall that an optimization problem A is called NP-hard if some NP-complete language can be recognized in polynomial time with the aid of an oracle for A . We assume that $P \neq NP$.

Definition 2.1. Suppose we are given a finite set A and a constraint language $\Gamma \subseteq \bigcup_{k=1}^{\infty} 2^{A^k}$. The language Γ is said to be *tractable* if, for every finite subset $\Gamma' \subseteq \Gamma$, $\text{MinHom}(\Gamma')$ is

polynomial-time solvable, and Γ is called *NP-hard* if there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\text{MinHom}(\Gamma')$ is NP-hard.

First, we will state some standard definitions from universal algebra.

Definition 2.2. Let $\rho \subseteq A^m$ and $f : A^n \rightarrow A$. We say that the function (operation) f *preserves* the predicate ρ if, for every $(x_1^i, \dots, x_m^i) \in \rho, 1 \leq i \leq n$, we have that $(f(x_1^1, \dots, x_1^n), \dots, f(x_m^1, \dots, x_m^n)) \in \rho$.

For a constraint language Γ , let $\text{Pol}(\Gamma)$ denote the set of operations preserving all predicates in Γ . Throughout the paper, we let A denote a finite domain and Γ a constraint language over A . We assume the domain A to be finite.

Definition 2.3. A constraint language Γ is called a *relational clone* if it contains every predicate expressible by a first-order formula involving only

- predicates from $\Gamma \cup \{=^A\}$;
- conjunction; and
- existential quantification.

First-order formulas involving only conjunction and existential quantification are often called *primitive positive (pp) formulas*. For a given constraint language Γ , the set of all predicates that can be described by pp-formulas over Γ is called the *closure* of Γ and is denoted by $\langle \Gamma \rangle$.

For a set of operations F on A , let $\text{Inv}(F)$ denote the set of predicates preserved under the operations of F . Obviously, $\text{Inv}(F)$ is a relational clone. The next result is well-known [3, 9].

Theorem 2.4. For a constraint language Γ over a finite set A , $\langle \Gamma \rangle = \text{Inv}(\text{Pol}(\Gamma))$.

Theorem 2.4 tells us that the Galois closure of a constraint language Γ is equal to the set of all predicates that can be obtained via pp-formulas from the predicates in Γ .

Theorem 2.5. For any finite constraint language Γ and any finite $\Gamma' \subseteq \langle \Gamma \rangle$, there is a polynomial time reduction from $\text{MinHom}(\Gamma')$ to $\text{MinHom}(\Gamma)$.

Proof. Since any predicate from Γ' can be viewed as a pp-formula with predicates in Γ , an input formula to $\text{MinHom}(\Gamma')$ can be represented on the form $\Phi(x_1, \dots, x_n) = \bigwedge_{i=1}^N \exists z_{i1}, \dots, z_{im_i} \Phi_i(y_{i1}, \dots, y_{in_i}, z_{i1}, \dots, z_{im_i})$, where $y_{ij} \in \{x_1, \dots, x_n\}$ and Φ_i is a first-order formula involving only predicates in Γ , equality, and conjunction. Obviously, this formula is equivalent to $\exists z_{11}, \dots, z_{Nm_N} \bigwedge_{i=1}^N \Phi_i(y_{i1}, \dots, y_{in_i}, z_{i1}, \dots, z_{im_i})$.

$\bigwedge_{i=1}^N \Phi_i(y_{i1}, \dots, y_{in_i}, z_{i1}, \dots, z_{im_i})$ can be considered as an instance of $\text{MinHom}(\Gamma \cup \{=^A\})$ with variables $x_1, \dots, x_n, z_{11}, \dots, z_{Nm_N}$ where weights w_{ij} will remain the same and for additional variables z_{kl} we define $w_{z_{kl}j} = 0$. By solving $\text{MinHom}(\Gamma \cup \{=^A\})$ with the described input, we can find a solution of the initial $\text{MinHom}(\Gamma')$ problem. It is easy to see that the number of added variables is bounded by a polynomial in n . So this reduction can be carried out in polynomial time. Finally, $\text{MinHom}(\Gamma \cup \{=^A\})$ can be reduced polynomially to $\text{MinHom}(\Gamma)$ because an equality constraint for a pair of variables is equivalent to identification of these variables. ■

The previous theorem tells us that the complexity of $MinHom(\Gamma)$ is basically determined by $Inv(Pol(\Gamma))$, i.e., by $Pol(\Gamma)$. That is why we will be concerned with the classification of sets of operations F for which $Inv(F)$ is a tractable constraint language.

Definition 2.6. An *algebra* is an ordered pair $\mathbb{A} = (A, F)$ such that A is a nonempty set (called a universe) and F is a family of finitary operations on A . An algebra with a finite universe is referred to as a finite algebra.

Definition 2.7. An algebra $\mathbb{A} = (A, F)$ is called *tractable* if $Inv(F)$ is a tractable constraint language and \mathbb{A} is called *NP-hard* if $Inv(F)$ is an NP-hard constraint language.

In the following theorem, we show that we only need to consider a very special type of algebras, so called *conservative* algebras.

Definition 2.8. An algebra $\mathbb{A} = (A, F)$ is called *conservative* if for every operation $f \in F$ we have that $f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$.

Theorem 2.9. For any finite constraint language Γ over A and $C \subseteq A$, there is a polynomial time Turing reduction from $MinHom(\Gamma \cup \{C\})$ to $MinHom(\Gamma)$.

Proof. Let the first-order formula $\Phi(x_1, \dots, x_n) = \bigwedge_{i=1}^M C(y_i) \wedge \bigwedge_{i=1}^N \rho_i(z_{i1}, \dots, z_{in_i})$, where $\rho_i \in \Gamma, y_i, z_{ij} \in \{x_1, \dots, x_n\}$, and weights $w_{ia}, 1 \leq i \leq n, a \in A$ be an instance of $MinHom(\Gamma \cup \{C\})$. We assume without loss of generality that $y_i \neq y_j$, when $i \neq j$.

Let $W = \sum_{i=1}^n \sum_{a \in A} w_{ia} + 1$ and define a new formula and weights

$$\Phi'(x_1, \dots, x_n) = \bigwedge_{i=1}^N \rho_i(z_{i1}, \dots, z_{in_i})$$

$$w'_{ia} = \begin{cases} w_{ia} + W, & \text{if } a \notin C, \exists j \ x_i = y_j \\ w_{ia}, & \text{otherwise} \end{cases}$$

Then, using an oracle for $MinHom(\Gamma)$, we can solve

$$\min_{f \text{ satisfies } \Phi'} \sum_j w'_{jf(x_j)}.$$

Suppose that $\Phi(x_1, \dots, x_n)$ is satisfiable and f is a satisfying assignment. It is easy to see that the part of the measure $\sum_j w'_{jf(x_j)}$ that corresponds to the added values W is equal to 0 and the measure cannot be greater than $W - 1$. If g is any assignment that does not satisfy $\bigwedge_{i=1}^M C(y_i)$, then we see that this part of measure cannot be 0, and hence, is greater or equal to W . This means that the minimum in the task is achieved on satisfying assignments of $\Phi(x_1, \dots, x_n)$ and any such assignment minimize the part of the measure that corresponds to the initial weights, i.e., $\sum_i w_{if(x_i)}$.

If $\Phi(x_1, \dots, x_n)$ is not satisfiable, then either Φ' is not satisfiable or $\min_{f \text{ satisfies } \Phi'} \sum_j w'_{jf(x_j)} \geq W$. Using an oracle for $MinHom(\Gamma)$, we can easily check this.

Consequently, $MinHom(\Gamma \cup \{C\})$ is polynomial-time reducible to $MinHom(\Gamma)$. ■

Theorem 2.10. If Γ is a constraint language over A that contains all unary relations, then $\mathbb{A} = (A, Pol(\Gamma))$ is conservative.

Proof. Let $C = \{x_1, \dots, x_n\} \subseteq A$. If a function $f : A^n \rightarrow A$ preserves the predicate C , then $f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$. ■

3. Structure of tractable conservative algebras

Let $g : A^k \rightarrow A$ be an arbitrary conservative function and $S \subseteq A$. Define the function $g|_S : S^k \rightarrow S$, such that $\forall x_1, \dots, x_k \in S \ g|_S(x_1, \dots, x_k) = g(x_1, \dots, x_k)$, i.e. the restriction of g to the set S . Throughout this paper we will consider a conservative algebra $\mathbb{A} = (A, F)$. For every $B \subseteq A$, let $F|_B = \{f_B | f \in F\}$. Then $\mathbb{A}|_B$ denotes an algebra (B, F_B) . We assume that F is closed under superposition and variable change and contains all projections, i.e., it is a *functional clone*, because closing the set F under these operations does not change the set $\text{Inv}(F)$.

Sometimes we will consider clones as algebras and to describe them we will use the terms (conservativeness, tractability, NP-hardness) defined for algebras. All tractable clones, in case $A = \{0, 1\}$, can be easily found using well-known classification of boolean clones [23].

Theorem 3.1. *The boolean functional clone H is tractable if either $\{x \wedge y, x \vee y\} \subseteq H$ or $\{(x \wedge \bar{y}) \vee (\bar{y} \wedge z) \vee (x \wedge z)\} \subseteq H$, where \wedge, \vee denote conjunction and disjunction. Otherwise, H is NP-hard.*

In the proof of this theorem we will need the following definition.

Definition 3.2. A constraint language Γ over A is called a *maximal tractable* constraint language if it is tractable and is not contained in any other tractable languages.

Let us identify all maximal tractable constraint languages in the boolean case using Post's classification [23]. From Theorems 2.5, 2.9, 2.10 we conclude that every maximal tractable constraint language corresponds to some conservative functional clone. In the case $A = \{0, 1\}$, there is a countable number of conservative clones: we list them below according to the table on page 76 [22]. For every row, the closure of the predicates given is equal to the set of all predicates preserved under the functions of the corresponding clone.

T_{01}	$x = 0, x = 1$
M_{01}	$x = 0, x = 1, x_1 \leq x_2$
S_{01}	$x = 0, x_1 \neq x_2$
SM	$x_1 \neq x_2, x_1 \leq x_2$
L_{01}	$x = 1, x_1 \oplus x_2 \oplus x_3 = 0$
U_{01}	$x = 0, x = 1, x_1 = x_2 \vee x_1 = x_3$
K_{01}	$x = 0, x = 1, x_1 = x_2 x_3$
D_{01}	$x = 0, x = 1, x_1 = x_2 \vee x_3$
I_1^m	$x = 1, x_1 x_2 \dots x_m = 0$
MI_1^m	$x = 1, x_1 \leq x_2, x_1 x_2 \dots x_m = 0$
O_0^m	$x = 0, x_1 \vee x_2 \vee \dots \vee x_m = 1$
MO_0^m	$x = 0, x_1 \leq x_2, x_1 \vee x_2 \vee \dots \vee x_m = 1$

where $x \oplus y = x + y \pmod{2}$.

Lemma 3.3. *The relational clones $\text{Inv}(M_{01})$ and $\text{Inv}(S_{01})$ are maximal tractable constraint languages. Every other constraint language given in the table, except $\text{Inv}(T_{01})$, is NP-hard.*

Proof. The class $\text{Inv}(T_{01})$ is tractable, since it contains only two simple unary predicates $\{0\}$ and $\{1\}$. As we will see later, it cannot be maximal since it is included in other tractable constraint languages.

Let us prove that $\text{Inv}(M_{01})$ is tractable. By Theorem 2.5, it is equivalent to polynomial solvability of $\text{MinHom}(\{\{0\}, \{1\}, \{(x_1, x_2) | x_1 \leq x_2\}\})$, because the class $\text{Inv}(M_{01})$ is the

closure of this set of predicates. A proof of this statement can be found in [19]. We will give it for completeness.

Obviously, $MinHom(\{\{0\}, \{1\}, \{(x_1, x_2) | x_1 \leq x_2\}\})$ is equivalent to the following boolean linear programming task, sets $Q_0, Q_1 \subseteq \{1, \dots, n\}, Q \subseteq \{1, \dots, n\}^2$ and integer weights w_1, \dots, w_n given as an input:

$$\begin{cases} \min \sum_i w_i x_i \\ x_i = 0, i \in Q_0 \\ x_i = 1, i \in Q_1 \\ x_i \leq x_j, (i, j) \in Q \\ x_i \in \{0, 1\} \end{cases}$$

Let us prove that the polyhedron which is given by the same equalities and inequalities as previous, but with $x_i \in \{0, 1\}$ replaced by $0 \leq x_i \leq 1$, is integer. Suppose it is not integer and $v = ||v_1, v_2, \dots, v_n||^T$ is its extreme point where v_r is not equal to 0 or 1. Let us define ϵ as the minimum of three values $\min_{v_i \neq v_j} |v_i - v_j|$, $\min_{v_i \neq 0} |v_i|$, $\min_{v_i \neq 1} |1 - v_i|$ and two vectors v^+ and v^- : $v_i^+ = v_i^- = v_i$ if $v_i \neq v_r$ and $v_i^+ = v_i + \epsilon$, $v_i^- = v_i - \epsilon$, otherwise. It is easy to see that points v^+ and v^- are also in polyhedron, and $v = \frac{v^+ + v^-}{2}$. This contradicts the extremeness of v .

Since the polyhedron is integer we can solve $MinHom(\{\{0\}, \{1\}, \{(x_1, x_2) | x_1 \leq x_2\}\})$ in polynomial time by standard linear programming algorithms. Consequently, $Inv(M_{01})$ is tractable.

Now let us prove that $Inv(S_{01})$ is tractable, i.e. $MinHom(\{\{0\}, \{(x_1, x_2) | x_1 \neq x_2\}\})$ is polynomial-time solvable.

Let an instance of this problem be the sets $Q_0 \subseteq \{1, \dots, n\}, Q \subseteq \{1, \dots, n\}^2$ and integer weights $w_{10}, \dots, w_{n0}, w_{11}, \dots, w_{n1}$. By $\Phi(Q_0, Q)$ we denote the set of assignments of variables x_1, \dots, x_n that satisfy the input formula, i.e. such that $x_i = 0, i \in Q_0$ and $x_k \neq x_l, (k, l) \in Q$.

The graph $(\{1, \dots, n\}, Q')$ where $Q' = \{(x, y) | (x, y) \in Q \vee (y, x) \in Q\}$ can be decomposed into connected components $(\{1, \dots, n\}, Q') = K_1 \cup \dots \cup K_t$, where $K_i = (V_i, E_i)$. Such a decomposition can be made in $O(n^2)$ steps. If among these components there is a graph with an odd cycle, then, obviously, $\Phi(Q_0, Q) = \emptyset$. Otherwise, the optimization task can be reduced to subtasks for every component. I.e., if for some component $\Phi(Q_0 \cap V_i, E_i) = \emptyset$, then $\Phi(Q_0, Q) = \emptyset$, otherwise:

$$\min_{\bar{x} \in \Phi(Q_0, Q)} \sum_{i=1}^n w_i x_i = \sum_{i=1}^t \min_{\bar{x} \in \Phi(Q_0 \cap V_i, E_i)} \sum_{j \in V_i} w_j x_j.$$

But $|\Phi(Q_0 \cap V_i, E_i)| \leq 2$, and a straightforward algorithm solves every subtask. So, $Inv(S_{01})$ is tractable.

We first now show that the classes in the table, except $Inv(M_{01})$, $Inv(S_{01})$ and $Inv(T_{01})$, are NP-hard. Since,

$$\begin{aligned} x_1 \vee x_2 &= \exists x_3 [x_1 \neq x_3] \wedge [x_3 \leq x_2] \\ x_1 \vee x_2 &= \exists x_3 [x_3 = 1] \wedge [x_3 = x_1 \vee x_3 = x_2] \\ \overline{x_1} \vee \overline{x_2} &= \exists x_3 [x_3 = 0] \wedge [x_3 = x_1 x_2] \\ x_1 \vee x_2 &= \exists x_3 [x_3 = 1] \wedge [x_3 = x_1 \vee x_2] \\ \overline{x_1} \vee \overline{x_2} &= \exists x_3 \dots x_m [x_1 x_2 \dots x_m = 0] \wedge [x_2 = x_3] \wedge \dots \wedge [x_{m-1} = x_m] \\ x_1 \vee x_2 &= \exists x_3 \dots x_m [x_1 \vee x_2 \vee \dots \vee x_m = 1] \wedge [x_2 = x_3] \wedge \dots \wedge [x_{m-1} = x_m] \end{aligned}$$

we see that $\{(x_1, x_2) | x_1 \vee x_2\} \in Inv(SM), Inv(U_{01}), Inv(D_{01}), Inv(O_0^m), Inv(MO_0^m)$ and $\{(x_1, x_2) | \overline{x_1} \vee \overline{x_2}\} \in Inv(K_{01}), Inv(I_1^m), Inv(MI_1^m)$.

We first prove that $MinHom(\{(x_1, x_2) | x_1 \vee x_2\})$ is NP-hard. Suppose an instance of this problem consists of an undirected graph $G = (V, E)$ where each vertex is considered as a variable. For each pair of variables $(u, v) \in E$, we require their assignments to satisfy $u = 1$ or $v = 1$. It is easy to see that for any such assignment f , the set $\{x | f(x) = 0\}$ is independent in the graph G . Furthermore, for any independent set S in the graph G , $g(x) = [x \notin S]$ is a satisfying assignment. If we define $w_{i0} = 1, w_{i1} = 1$ for $i \in V$, then $MinHom$ is equivalent to finding a maximum independent set. This implies that $MinHom(\{(x_1, x_2) | x_1 \vee x_2\})$ is NP-hard, since finding independent sets of maximal size is an NP-hard problem. The case $MinHom(\{(x_1, x_2) | \overline{x_1} \vee \overline{x_2}\})$ is analogous.

Therefore, $Inv(SM), Inv(U_{01}), Inv(D_{01}), Inv(O_0^m), Inv(MO_0^m), Inv(K_{01}), Inv(I_1^m), Inv(MI_1^m)$ are NP-hard, too.

It remains to prove NP-hardness of $Inv(L_{01})$. We show that using an algorithm for $MinHom(\{(x_1, x_2, x_3) | x_1 \oplus x_2 \oplus x_3 = 1\})$ as an oracle, we can solve Max-CUT in polynomial time.

Let $G = (V, E)$ be a graph and introduce variables $x_{ij}, y_i, y_j, i, j \in V$. A system of equations $x_{ij} \oplus y_i \oplus y_j = 1, i, j \in V$ can be viewed as an instance of $MinHom(\{(x_1, x_2, x_3) | x_1 \oplus x_2 \oplus x_3 = 1\})$. It is easy to see that arbitrary boolean vector $\overline{y} = (y_1, \dots, y_{|V|})$ defines a single solution $x_{ij} = y_i \oplus y_j \oplus 1, i, j \in V$ of the system. Vector \overline{y} can be considered as the cut $\{i | y_i = 1\} \subseteq V$ and the value $\sum_{ij} (1 - x_{ij})$ is equal to the doubled cost of the cut. Then Max-CUT is polynomially reduced to solving $MinHom(Inv(L_{01}))$.

Only two classes $Inv(M_{01})$ and $Inv(S_{01})$ are left as candidates for maximality. Since they are not included in each other, they are both maximal. ■

Lemma 3.4. *If a constraint language $S \subseteq \bigcup_{k=1}^{\infty} 2^{\{0,1\}^k}$ is contained in neither $Inv(M_{01})$ nor $Inv(S_{01})$, then it is NP-hard.*

Proof. Suppose we are given a constraint language S which is not contained in $Inv(M_{01})$ and $Inv(S_{01})$. Then, $\langle S \cup 2^A \rangle$ is not contained in $Inv(M_{01})$ and $Inv(S_{01})$, either. Since $\langle S \cup 2^A \rangle$ is a boolean conservative relational clone, then, by previous lemma, it is NP-hard. By Theorems 2.5 and 2.9, we conclude that S is NP-hard. ■

Proof of Theorem 3.1. The bases in the clones M_{01}, S_{01} are $\{\wedge, \vee\}$ and $\{(x \wedge \overline{y}) \vee (\overline{y} \wedge z) \vee (x \wedge z)\}$ and the theorem follows from Lemma 3.4. ■

Every 2-element subalgebra of a tractable algebra must be tractable, which motivates the following definition.

Definition 3.5. Let F be a conservative functional clone. We say that F satisfies the *necessary local conditions* if and only if for every 2-element subset $B \subseteq A$, either

- there exists $f^\wedge, f^\vee \in F$ s.t. $f^\wedge|_B$ and $f^\vee|_B$ are different binary commutative functions; or
- there exists $f \in F$ s.t. $f|_B(x, x, y) = f|_B(y, x, x) = f|_B(y, x, y) = y$.

Theorem 3.6. *Suppose F is a conservative functional clone. If F is tractable, then it satisfies the necessary local conditions. If F does not satisfy the necessary local conditions, then it is NP-hard.*

Proof. Since for every two-element subset $B \subseteq A$, $\text{Inv}(F|_B) \subseteq \text{Inv}(F)$, then $F|_B$ is tractable. Assume without loss of generality that $B = \{0, 1\}$. From Theorem 3.1, we get that $\{\wedge, \vee\} \subseteq F|_B$ or $\{a(x, y, z) = (x \wedge \bar{y}) \vee (\bar{y} \wedge z) \vee (x \wedge z)\} \subseteq F|_B$. \wedge, \vee is a pair of different commutative conservative functions and $a(x, x, y) = a(y, x, x) = a(y, x, y) = y$. ■

In general, the necessary local conditions are not sufficient for tractability of a conservative clone. Let $M = \{B|B \subseteq A, |B| = 2, F|_B \text{ contains different binary commutative functions}\}$ and $\bar{M} = \{B|B \subseteq A, |B| = 2\} \setminus M$.

Suppose $f \in F$. By $\overset{a}{\downarrow} f$ we mean $a \neq b$ and $f(a, b) = f(b, a) = b$. For example, $\overset{1}{\downarrow} \overset{2}{\downarrow} \overset{1}{\downarrow} f$ means that $f|_{\{1,2,3\}}(x, y) = \max(x, y)$.

Introduce an undirected graph without loops $T_F = (M^o, P)$ where $M^o = \{(a, b) | \{a, b\} \in M\}$ and $P = \left\{ \langle (a, b), (c, d) \rangle | (a, b), (c, d) \in M^o, \text{ there is no } f \in F : \overset{a}{\downarrow} \overset{c}{\downarrow} f \right\}$. The core result of the paper is the following.

Theorem 3.7. *Suppose F satisfy the necessary local conditions. If the graph $T_F = (M^o, P)$ is bipartite, then F is tractable. Otherwise, F is NP-hard.*

The proof of this theorem will be given in two steps. Firstly, in the following section, we will prove NP-hardness of F when $T_F = (M^o, P)$ is not bipartite. The final sections will be dedicated to the polynomial-time solvable cases.

4. NP-hard case

In this section, we will prove that if a set of functions F satisfies the necessary local conditions and $T_F = (M^o, P)$ (as defined in the previous section) is not bipartite, then F is NP-hard. Let $\overset{a}{\times}_b^c$ and $\overset{a}{\times}_b^c$ denote the predicates $\{a, b\} \times \{c, d\} \setminus \{(b, d)\}$ and $\{(a, d), (b, c)\}$, where $a \neq b, c \neq d$. We need the following lemmas.

Lemma 4.1. *A constraint language that contains $\left\{ \overset{a_0}{\times}_{b_0}^{a_1}, \dots, \overset{a_{2k-1}}{\times}_{b_{2k-1}}^{a_{2k}}, \overset{a_{2k}}{\times}_{b_{2k}}^{a_0} \right\}$ is NP-hard.*

Before proving Lemma 4.1, we need to introduce some concepts from graph theory. All graphs are assumed to be undirected and without loops. We will be interested in the complexity of finding independent sets of maximal size in classes of graphs. Let a finite

number of graphs G_1, \dots, G_k be given and let $\text{Free}(G_1, \dots, G_k)$ denote the set of graphs that has no induced subgraphs isomorphic to one of G_1, \dots, G_k .

The following theorem has been proved by Alekseev[1].

Theorem 4.2. *If there is no graph among G_1, \dots, G_k whose every connected component is a tree with at most 3 leaves, then the maximum independent set problem is NP-hard when restricted to graphs in $\text{Free}(G_1, \dots, G_k)$.*

Definition 4.3. The graph $G = (V, E)$ is said to be *homomorphic* to the graph $H = (W, S)$ if there is a mapping $f : V \rightarrow W$ such that $\forall (x, y) \in E \ (f(x), f(y)) \in S$. The mapping f is called an *H-homomorphism*.

Let C_d be a cycle of length d .

Theorem 4.4. *If $d \geq 3$ is odd, then the problem of finding a maximum independent set in an undirected graph homomorphic to C_d is NP-hard even if a C_d -homomorphism is given.*

Proof. First, we will prove NP-hardness of finding maximum independent sets in a graph homomorphic to C_3 , i.e. three-partite graph, following [13]. An instance consists of a graph and a partitioning into three independent sets.

Let X be a class of graphs with degrees not greater than 3. This class can be characterized by forbidden subgraphs — it is sufficient to forbid graphs with 5 vertices that has a vertex connected with 4 others. Obviously, every such graph is connected and if it is a tree it has 4 leaves. By Theorem 4.2 we conclude that finding maximum independent sets is NP-hard in the class X .

From Brooks' theorem[4], we have that every graph in X , besides the complete graph on 4 vertices, is three-partite. The required partition can be constructed in polynomial time by an algorithm of Lovasz[21]. Therefore, the problem of finding maximum independent sets in a three-partite graph is NP-hard even if a partition is given.

The case when $d = 3$ can be reduced to every odd case $d > 3$. Let a three-partite graph be given. We will define it in the following form: $G = (V_1, V_2, V_3, E_{12}, E_{23}, E_{31})$, where $E_{12} \subseteq V_1 \times V_2, E_{23} \subseteq V_2 \times V_3, E_{31} \subseteq V_3 \times V_1$. Transform G as follows: for each edge $(u, v) \in E_{12}$, add vertices $x_{uv1}, x_{uv2}, \dots, x_{uv(d-3)}$ to the graph, delete the edge (u, v) , and add edges $(u, x_{uv1}), (x_{uv1}, x_{uv2}), \dots, (x_{uv(d-3)}, v)$. The obtained graph G^d is, obviously, homomorphic to C_d .

Let n, N denote the independence numbers of G and G^d respectively. It is easy to see that $N \geq n + \frac{d-3}{2} |E_{12}|$. We prove that we actually have equality there. Note that intersection of any maximum independent set of G^d and $\{u, x_{uv1}, x_{uv2}, \dots, x_{uv(d-3)}, v\}$ contains not less than $\frac{d-3}{2}$, and not more than $\frac{d-1}{2}$ elements. In the first case ($\frac{d-3}{2}$), we can delete all elements $u, x_{uv1}, x_{uv2}, \dots, x_{uv(d-3)}, v$ from the independent set and replace them by $x_{uv1}, x_{uv3}, x_{uv5}, \dots, x_{uv(d-4)}$, while not destroying independency. In the second case ($\frac{d-1}{2}$), either u or v are always in the independent set. Again, we delete $u, x_{uv1}, x_{uv2}, \dots, x_{uv(d-3)}, v$ from it. In the case where u was in the independent set originally, we replace the deleted elements by $\{u, x_{uv2}, x_{uv4}, \dots, x_{uv(d-3)}\}$ and otherwise by $\{x_{uv1}, x_{uv3}, \dots, x_{uv(d-4)}, v\}$. As a result, we obtain independent set of G^d with the same cardinality as initially. This operation can be done with all pairs $uv \in E_{12}$. It is easy to see that intersection of the obtained set with $V_1 \cup V_2 \cup V_3$ is an independent set in G and it has cardinality $N - \frac{d-3}{2} |E_{12}|$. Consequently, $N = n + \frac{d-3}{2} |E_{12}|$ and the constructed intersection is a maximum independent set in G . The steps of construction can be carried in polynomial time. Thus, by finding a

maximum independent set in G^d , we can easily reconstruct that of G . This means that the maximum independent set problem in a three-partite graph is polynomial-time reducible to the maximum independent set problem in a graph homomorphic to C_d (with given homomorphism). \blacksquare

Proof of Lemma 4.1. We show that finding a maximum independent set in a graph homomorphic to C_{2k+1} can be reduced to $MinHom(\Gamma)$ where $\Gamma = \left\{ \begin{smallmatrix} a_0 \diagdown a_1 \\ b_0 \diagup b_1 \end{smallmatrix}, \begin{smallmatrix} a_1 \diagdown a_2 \\ b_1 \diagup b_2 \end{smallmatrix}, \dots, \begin{smallmatrix} a_{2k-1} \diagdown a_{2k} \\ b_{2k-1} \diagup b_{2k} \end{smallmatrix}, \begin{smallmatrix} a_{2k} \diagdown a_0 \\ b_{2k} \diagup b_0 \end{smallmatrix} \right\}$.

Suppose the task is to find a maximum independent set in a graph homomorphic to C_{2k+1} , which, for convenience, will be given in the following form: $G = (V_0, V_1, \dots, V_{2k}, E_{i,i \oplus 1} \subseteq V_i \times V_{i \oplus 1})$, where $i \oplus j$ denotes $i + j \pmod{2k+1}$. We consider every vertex $v \in \bigcup_{i=0}^{2k} V_i$ as a variable and require values of variables $(u, v) \in V_i \times V_{i \oplus 1}$ to satisfy the constraint $\begin{smallmatrix} a_i \diagdown a_{i \oplus 1} \\ b_i \diagup b_{i \oplus 1} \end{smallmatrix}$. The set of satisfying assignments is denoted by Φ . It is easy to see that

$$\Phi = \left\{ f \mid \forall v \in V_i \ f(v) \in \{a_i, b_i\}, \bigcup_i \{x \mid x \in V_i, f(x) = b_i\} - \text{independent set in } G \right\}.$$

Therefore, the task

$$\min_{f \in \Phi} \sum_i \sum_{x \in V_i} [f(x) \neq b_i]$$

is equivalent to finding a maximum independent set in the graph G . I.e., it is equivalent to the $MinHom(H)$ problem with an instance consisting of the defined constraints on the variables $\bigcup_{i=0}^{2k} V_i$ and weights $w_{xa_i} = 1, w_{xb_i} = 0$. Consequently, $MinHom(H)$ is NP-hard. \blacksquare

Lemma 4.5. *If $\langle (a, b), (c, d) \rangle \in P$, then either $\begin{smallmatrix} a \diagdown c \\ b \diagup d \end{smallmatrix} \in Inv(F)$, or $\begin{smallmatrix} a \diagup c \\ b \diagdown d \end{smallmatrix} \in Inv(F)$.*

Proof. We begin by constructing functions $\phi_1, \phi_2 \in F$ such that $\begin{smallmatrix} a \ c \\ b \ d \end{smallmatrix} \downarrow \uparrow \phi_1, \begin{smallmatrix} a \ c \\ b \ d \end{smallmatrix} \uparrow \downarrow \phi_2$. The symbol

$\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \downarrow \uparrow \lambda$ means that either $\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \uparrow \lambda$, or $\lambda|_{\{\alpha, \beta\}}$ is a projection.

Since $\{a, b\}, \{c, d\} \in M$, we have $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in F : \begin{smallmatrix} a \\ b \end{smallmatrix} \downarrow \lambda_1, \begin{smallmatrix} a \\ b \end{smallmatrix} \uparrow \lambda_2, \begin{smallmatrix} c \\ d \end{smallmatrix} \downarrow \lambda_3, \begin{smallmatrix} c \\ d \end{smallmatrix} \uparrow \lambda_4$.

Moreover, by the definition of P , we have $\begin{smallmatrix} c \\ d \end{smallmatrix} \downarrow \lambda_1, \begin{smallmatrix} a \\ b \end{smallmatrix} \downarrow \lambda_3$. By defining $\phi_1(x, y) =$

$\lambda_4(\lambda_1(x, y), \lambda_1(y, x)), \phi_2(x, y) = \lambda_2(\lambda_3(x, y), \lambda_3(y, x)) \in F$, we see that $\begin{smallmatrix} a \ c \\ b \ d \end{smallmatrix} \downarrow \uparrow \phi_1, \begin{smallmatrix} a \ c \\ b \ d \end{smallmatrix} \uparrow \downarrow \phi_2$.

Suppose $\begin{smallmatrix} a \diagdown c \\ b \diagup d \end{smallmatrix} \notin Inv(F)$. We prove that in this case $\begin{smallmatrix} a \diagup c \\ b \diagdown d \end{smallmatrix} \in Inv(F)$. Since the predicate $\begin{smallmatrix} a \diagdown c \\ b \diagup d \end{smallmatrix}$ consists of three pairs, it is not preserved by some function of arity two or three. Let us consider these two cases:

I. A function $\phi \in F$ of arity two does not preserve $\begin{smallmatrix} a \diagdown c \\ b \diagup d \end{smallmatrix}$ if (for some appropriate permutation of variables):

$$\begin{aligned} \phi(a, b) &= b \\ \phi(d, c) &= d \end{aligned}$$

Then $\begin{smallmatrix} a \ c \\ b \ d \end{smallmatrix} \downarrow \phi(\phi_2(x, y), \phi_1(x, y))$ which contradicts that $\langle (a, b), (c, d) \rangle \in P$.

II. A function $\phi \in F$ of arity three does not preserve $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \times_d$ if (for some appropriate permutation of variables):

$$\begin{aligned}\phi(a, a, b) &= b \\ \phi(d, c, c) &= d\end{aligned}$$

Then, $\langle (b, a), (d, c) \rangle \in P$, since, otherwise, we can find $\phi_3 \in F : \begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \uparrow \uparrow \phi_3$ and construct the

following term $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \downarrow \downarrow \phi(\phi_2(x, y), \phi_3(x, y), \phi_1(x, y))$. This contradicts that $\langle (a, b), (c, d) \rangle \in P$.

Suppose instead that $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \times_d \notin \text{Inv}(F)$, i.e., there is a function $f \in F$ of arity two that does not preserve $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \times_d$. If f does not preserve $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \times_d$, then it does not preserve either $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \times_d^c$, or $\begin{smallmatrix} b & d \\ a & c \end{smallmatrix} \times_c^d$. Since $\langle (a, b), (c, d) \rangle, \langle (b, a), (d, c) \rangle \in P$, we get a contradiction in both cases via the same argument as in case I. \blacksquare

Proof of NP-hard case of Theorem 3.7. For binary predicates α, β , let $\alpha \circ \beta = \{(x, y) \mid \exists z : \alpha(x, z) \wedge \beta(z, y)\}$. Obviously, if $\alpha, \beta \in \text{Inv}(F)$, then $\alpha \circ \beta \in \text{Inv}(F)$, too.

Since $T_F = (M^o, P)$ is not bipartite, we can find a shortest odd cycle in it, i.e. a sequence $(a_0, b_0), (a_1, b_1), \dots, (a_{2k}, b_{2k}) \in M^o, k \geq 1$, such that $\langle (a_i, b_i), (a_{i \oplus 1}, b_{i \oplus 1}) \rangle \in P$. Here, $i \oplus j$ denotes $i + j \pmod{2k+1}$.

By Lemma 4.5, there is a cyclic sequence $\rho_{0,1}, \rho_{1,2}, \dots, \rho_{2k,0} \in \text{Inv}(F)$ such that $\rho_{i,i \oplus 1}$ is either equal to $\begin{smallmatrix} a_i & a_{i \oplus 1} \\ b_i & b_{i \oplus 1} \end{smallmatrix} \times_{b_{i \oplus 1}}$ or equal to $\begin{smallmatrix} a_i & a_{i \oplus 1} \\ b_i & b_{i \oplus 1} \end{smallmatrix} \times_{b_{i \oplus 1}}$. Note that all predicates cannot be of the second type: otherwise, we have $\rho_{0,1} \circ \rho_{1,2} \circ \dots \circ \rho_{2k,0} = \begin{smallmatrix} a_0 & a_0 \\ b_0 & b_0 \end{smallmatrix} \times_{b_0}$ which contradicts that $\{a_0, b_0\} \in M$.

If the sequence contains a fragment $\rho_{i,i \oplus 1} = \begin{smallmatrix} a_i & a_{i \oplus 1} \\ b_i & b_{i \oplus 1} \end{smallmatrix} \times_{b_{i \oplus 1}}$, $\rho_{i \oplus 1, i \oplus 2} = \begin{smallmatrix} a_{i \oplus 1} & a_{i \oplus 2} \\ b_{i \oplus 1} & b_{i \oplus 2} \end{smallmatrix} \times_{b_{i \oplus 2}}$, $\rho_{i \oplus 2, i \oplus 3} = \begin{smallmatrix} a_{i \oplus 2} & a_{i \oplus 3} \\ b_{i \oplus 2} & b_{i \oplus 3} \end{smallmatrix} \times_{b_{i \oplus 3}}$, then these predicates can be replaced by:

$$\rho_{i,i \oplus 3} \stackrel{\Delta}{=} \rho_{i,i \oplus 1} \circ \rho_{i \oplus 1, i \oplus 2} \circ \rho_{i \oplus 2, i \oplus 3} = \begin{smallmatrix} a_i & a_{i \oplus 1} \\ b_i & b_{i \oplus 1} \end{smallmatrix} \times_{b_{i \oplus 1}} \circ \begin{smallmatrix} a_{i \oplus 1} & a_{i \oplus 2} \\ b_{i \oplus 1} & b_{i \oplus 2} \end{smallmatrix} \times_{b_{i \oplus 2}} \circ \begin{smallmatrix} a_{i \oplus 2} & a_{i \oplus 3} \\ b_{i \oplus 2} & b_{i \oplus 3} \end{smallmatrix} \times_{b_{i \oplus 3}} = \begin{smallmatrix} a_i & a_{i \oplus 3} \\ b_i & b_{i \oplus 3} \end{smallmatrix} \times_{b_{i \oplus 3}}$$

Let us replace $\rho_{i,i \oplus 1}, \rho_{i \oplus 1, i \oplus 2}, \rho_{i \oplus 2, i \oplus 3}$ by $\rho_{i,i \oplus 3}$ in the sequence $\rho_{0,1}, \rho_{1,2}, \dots, \rho_{2k,0}$. We have $\langle (a_i, b_i), (a_{i \oplus 3}, b_{i \oplus 3}) \rangle \in P$, since otherwise the predicate $\rho_{i,i \oplus 3}$ is not preserved. Hence, we can delete two vertices in the cycle $(a_0, b_0), (a_1, b_1), \dots, (a_{2k}, b_{2k}) \in M^o$. This contradicts that this sequence is the shortest among odd sequences. Therefore, such a fragment does not exist.

If the sequence contains a fragment $\rho_{i,i \oplus 1} = \begin{smallmatrix} a_i & a_{i \oplus 1} \\ b_i & b_{i \oplus 1} \end{smallmatrix} \times_{b_{i \oplus 1}}$, $\rho_{i \oplus 1, i \oplus 2} = \begin{smallmatrix} a_{i \oplus 1} & a_{i \oplus 2} \\ b_{i \oplus 1} & b_{i \oplus 2} \end{smallmatrix} \times_{b_{i \oplus 2}}$, $\rho_{i \oplus 2, i \oplus 3} = \begin{smallmatrix} a_{i \oplus 2} & a_{i \oplus 3} \\ b_{i \oplus 2} & b_{i \oplus 3} \end{smallmatrix} \times_{b_{i \oplus 3}}$, then these predicates can be replaced by:

$$\rho_{i,i \oplus 3} \stackrel{\Delta}{=} \rho_{i,i \oplus 1} \circ \rho_{i \oplus 1, i \oplus 2} \circ \rho_{i \oplus 2, i \oplus 3} = \begin{smallmatrix} a_i & a_{i \oplus 1} \\ b_i & b_{i \oplus 1} \end{smallmatrix} \times_{b_{i \oplus 1}} \circ \begin{smallmatrix} a_{i \oplus 1} & a_{i \oplus 2} \\ b_{i \oplus 1} & b_{i \oplus 2} \end{smallmatrix} \times_{b_{i \oplus 2}} \circ \begin{smallmatrix} a_{i \oplus 2} & a_{i \oplus 3} \\ b_{i \oplus 2} & b_{i \oplus 3} \end{smallmatrix} \times_{b_{i \oplus 3}} = \begin{smallmatrix} a_i & a_{i \oplus 3} \\ b_i & b_{i \oplus 3} \end{smallmatrix} \times_{b_{i \oplus 3}}$$

As in the previous case, we obtain a contradiction. Consequently, we have an odd sequence $\begin{smallmatrix} a_0 & a_1 \\ b_0 & b_1 \end{smallmatrix} \times_{b_1}, \begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix} \times_{b_2}, \dots, \begin{smallmatrix} a_{2k-1} & a_{2k} \\ b_{2k-1} & b_{2k} \end{smallmatrix} \times_{b_{2k}}, \begin{smallmatrix} a_{2k} & a_0 \\ b_{2k} & b_0 \end{smallmatrix} \times_{b_0} \in \text{Inv}(F)$. By Lemma 4.1, this class of predicates is NP-hard. \blacksquare

5. Existence of the majority operation

The necessary local conditions tell that every two-element subalgebra of a tractable algebra contains certain operations. The simplest algebras over a domain A that satisfy these conditions are the following: $F_1 = \{\phi, \psi\}$ where ϕ, ψ are conservative commutative operations such that $\phi(a, b) \neq \psi(a, b)$ for every $a \neq b \in A$, and $F_2 = \{m\}$ where m is a

conservative arithmetical operation, i.e. $m(x, x, y) = m(y, x, x) = m(y, x, y) = y$. This leads us to the following definitions.

Definition 5.1. Suppose a set of operations H over D is conservative and $B \subseteq \{\{x, y\} \mid x, y \in D, x \neq y\}$. A pair of binary operations $\phi, \psi \in H$ is called a *tournament pair* on B , if $\forall \{x, y\} \in B$ $\phi(x, y) = \phi(y, x)$, $\psi(x, y) = \psi(y, x)$, $\phi(x, y) \neq \psi(x, y)$ and for arbitrary $\{x, y\} \in \overline{B}$, $\phi(x, y) = x$, $\psi(x, y) = x$. An operation $m \in H$ is called *arithmetical* on B , if $\forall \{x, y\} \in B$ $m(x, x, y) = m(y, x, x) = m(y, x, y) = y$.

Definition 5.2. An operation $\mu : A^3 \rightarrow A$, satisfying the equality

$$\mu(x, y, y) = \mu(y, x, y) = \mu(y, y, x) = y$$

is called a majority operation.

Theorem 5.3. If F satisfies the necessary local conditions and $T_F = (M^o, P)$ is bipartite, then F contains a tournament pair on M .

Proof. Let M_1, M_2 denote a partitioning of the bipartite graph $T_F = (M^o, P)$. Then, for every $(a, b), (c, d) \in M_1$, there is a function $\phi \in F : \begin{smallmatrix} a & c \\ \downarrow & \downarrow \\ b & d \end{smallmatrix} \phi$. Let us prove by induction that

for every $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in M_1$, there is a $\phi : \begin{smallmatrix} a_1 & a_2 & \dots & a_n \\ \downarrow & \downarrow & \dots & \downarrow \\ b_1 & b_2 & \dots & b_n \end{smallmatrix} \phi$.

The base of induction $n = 2$ is obvious. Let $(a_1, b_1), (a_2, b_2), \dots, (a_{n+1}, b_{n+1}) \in M_1$ be given. By the induction hypothesis, there are $\phi_1, \phi_2, \phi_3 \in F$:
 $\begin{smallmatrix} a_2 & a_n & a_{n+1} \\ \downarrow & \downarrow & \downarrow \\ b_2 & b_n & b_{n+1} \end{smallmatrix} \phi_1, \begin{smallmatrix} a_1 & a_3 & \dots & a_n & a_{n+1} \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow \\ b_1 & b_3 & \dots & b_n & b_{n+1} \end{smallmatrix} \phi_2, \begin{smallmatrix} a_1 & a_2 & \dots & a_n \\ \downarrow & \downarrow & \dots & \downarrow \\ b_1 & b_2 & \dots & b_n \end{smallmatrix} \phi_3$.
 $\begin{smallmatrix} a_1 & a_n & a_{n+1} \\ \downarrow & \downarrow & \downarrow \\ b_1 & b_n & b_{n+1} \end{smallmatrix} \phi_3(\phi_1(x, y), \phi_2(x, y))$ which completes the induction proof.

The analogous statement can be proved for M_2 . Moreover, $M_2 = \{(x, y) \mid (y, x) \in M_1\}$. So it follows from the proof that there are binary operations $\phi', \psi' \in F$, such that $\forall (x, y) \in M_1 : \begin{smallmatrix} x \\ \downarrow \\ y \end{smallmatrix} \phi'$ and $\forall (x, y) \in M_2 : \begin{smallmatrix} x \\ \downarrow \\ y \end{smallmatrix} \psi'$. Thus, the operations $\phi(x, y) = \phi'(x, \phi'(y, x))$ and $\psi(x, y) = \psi'(x, \psi'(y, x))$ satisfy the conditions of theorem. \blacksquare

The proof of the following theorem uses ideas from [5].

Theorem 5.4. If F satisfies the necessary local conditions and $\overline{M} \neq \emptyset$, then F contains an arithmetical operation on \overline{M} .

Proof. Note first that for every $B \in \overline{M}$, $F|_B$ cannot contain any commutative binary function. To see this, assume that $B = \{0, 1\}$ and note that $F|_B$ contains S_{01} and either conjunction or disjunction. From Post's results [23], we see that $F|_B$ contains all boolean functions preserving 0 and 1, i.e., contains both conjunction and disjunction. This contradicts that $B \notin M$. Therefore, every binary function in $F|_B$ is a projection.

For $B \in \overline{M}$, let m^B be an arithmetical function on B ; existence of this function follows from the necessary local conditions. Assume now that $\overline{M} = \{\{x_1, y_1\}, \dots, \{x_s, y_s\}\}$. We prove by induction that for every $r \leq s$, F contains a function $m_r : A^3 \rightarrow A$ that is arithmetical on $\{\{x_i, y_i\} \mid 1 \leq i \leq r\}$.

When $r = 1$, $m_1(x, y, z) = m^{\{x_1, y_1\}}(x, y, z)$ and the statement is obviously true. Suppose it is true for $r \leq k < s$ and that we have the function $m_k : A^3 \rightarrow A$. Let us prove the

statement for $r = k + 1$. If m_k is arithmetical on $\{\{x_{k+1}, y_{k+1}\}\}$, then we define $m_{k+1} = m_k$ and the statement is proved. Otherwise, one of the following three statements is true

$$\begin{aligned} \exists x, y \in \{x_{k+1}, y_{k+1}\} [m_k(x, x, y) \neq y], \\ \exists x, y \in \{x_{k+1}, y_{k+1}\} [m_k(y, x, x) \neq y], \\ \exists x, y \in \{x_{k+1}, y_{k+1}\} [m_k(y, x, y) \neq y]. \end{aligned}$$

Suppose the first case holds (the proof for other cases is analogous), i.e. $m_k|_{\{x_{k+1}, y_{k+1}\}}(x, x, y)$ is the x -projection. It is easy to see that the function $m_{k+1}(x, y, z) = m_k(m^{\{x_{k+1}, y_{k+1}\}}(x, y, z), m^{\{x_{k+1}, y_{k+1}\}}(x, y, z), m_k(x, y, z))$ is arithmetical on $\{\{x_i, y_i\} \mid 1 \leq i \leq k + 1\}$.

Induction completed and it is clear that $m_s(x, y, z)$ satisfies the condition of theorem. ■

Theorem 5.5. *If F satisfies the necessary local conditions and $T_F = (M^o, P)$ is bipartite, then F contains a majority operation μ .*

Proof. If $\overline{M} \neq \emptyset$, then by Theorem 5.4, F contains a function $m : A^3 \rightarrow A$ that is arithmetical on \overline{M} . Then the function $\mu^1(x, y, z) = m(x, m(x, y, z), z)$ satisfies the conditions $\forall \{x, y\} \in \overline{M} \mu^1(x, y, y) = \mu^1(y, x, y) = \mu^1(y, y, x) = y$. It is clear that, in the case where $M = \emptyset$, we can take μ^1 as majority μ .

If $M \neq \emptyset$, then by Theorem 5.3, there is a tournament pair $\phi, \psi : A^2 \rightarrow A$ on M . Then, the function $\mu^2(x, y, z) = \phi(\phi(\psi(x, y), \psi(y, z)), \psi(x, z))$ satisfies conditions $\forall \{x, y\} \in M \mu^2(x, y, y) = \mu^2(y, x, y) = \mu^2(y, y, x) = y$, and $\forall \{x, y, z\} \in \overline{M} \mu^2(x, y, z) = x$. If $\overline{M} = \emptyset$, then we can take μ^2 as the majority μ .

Finally, if $M, \overline{M} \neq \emptyset$, then $\mu(x, y, z) = \mu^1(\mu^2(x, y, z), \mu^2(y, z, x), \mu^2(z, x, y))$. ■

6. Consistency and microstructure graphs

Every predicate in $Inv(F)$, when F contains a majority operation, is equal to the join of its binary projections [2]. To prove Theorem 3.7, it is consequently sufficient to prove polynomial-time solvability of $MinHom(\Gamma)$ where $\Gamma = \{\rho \mid \rho \subseteq A^2, \rho \in Inv(F)\}$, i.e. the $MinHom$ problem restricted to binary constraint languages.

Definition 6.1. Suppose we are given a constraint language Γ over A . Denote by $2 - MinHom(\Gamma)$ the following minimization problem:

Instance: A finite set of variables $X = \{x_1, \dots, x_n\}$, a constraints pair (U, B) where $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$, $\rho_i, \rho_{kl} \in \Gamma$, and weights $w_{ia}, 1 \leq i \leq n, a \in A$.

Solution: Assignment $f : \{x_1, \dots, x_n\} \rightarrow A$, such that $\forall i f(x_i) \in \rho_i$ and $\forall k \neq l (f(x_k), f(x_l)) \in \rho_{kl}$.

Measure: $\sum_{i=1}^n w_{if(x_i)}$.

We suppose everywhere that $\rho_{kl} = \rho_{lk}^t$ (where $\rho^t = \{(b, a) \mid (a, b) \in \rho\}$). If $\rho_{kl} \neq \rho_{lk}^t$, then we can always define $\forall k \neq l \rho_{kl} := \rho_{kl} \cap \rho_{lk}^t$, which does not change the set $\{(a, b) \mid (a, b) \in \rho_{kl}, (b, a) \in \rho_{lk}\}$. For a binary predicate ρ , define projections $Pr_1 \rho = \{a \mid (a, b) \in \rho\}$ and $Pr_2 \rho = \{b \mid (a, b) \in \rho\}$.

Definition 6.2. An instance of $2 - MinHom(\Gamma)$ with constraints pair $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ is called *arc-consistent* if $\forall i \neq j : Pr_1 \rho_{ij} = \rho_i, Pr_2 \rho_{ij} = \rho_j$ and is called *path-consistent* if for each different $i, j, k : \rho_{ik} \subseteq \rho_{ij} \circ \rho_{jk}$.

Obviously, by applying operations $\rho_i := \rho_i \cap \text{Pr}_1 \rho_{ij}$, $\rho_j := \rho_j \cap \text{Pr}_2 \rho_{ij}$, $\rho_{ij} := \rho_{ij} \cap (\rho_i \times A)$, $\rho_{ij} := \rho_{ij} \cap (A \times \rho_j)$, $\rho_{ik} := \rho_{ik} \cap (\rho_{ij} \circ \rho_{jk})$, we can always make an instance arc-consistent and path-consistent in polynomial time. It is clear that under this transformations the set of feasible solutions does not change.

Definition 6.3. The *microstructure graph* [18] of an instance of $2 - \text{MinHom}(\Gamma)$ with constraints pair $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ is the graph $M_{U,B} = (V, E)$, where $V = \{(i, a) \mid 1 \leq i \leq n, a \in \rho_i\}$ and $E = \{((i, a), (j, b)) \mid i \neq j, (a, b) \in \rho_{ij}\}$.

Theorem 6.4. Let $I = (X, U, B, w)$ be a satisfiable instance of $2 - \text{MinHom}(\Gamma)$. Then there is a one-to-one correspondence between maximal-size cliques of $M_{U,B}$ and satisfying assignments of I .

Proof. The microstructure graph of an instance with constraints pair $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ is, obviously, n -partite, since $V = \bigcup_{i=1}^n \{i\} \times \rho_i$ and pairs $(i, a), (i, b), a \neq b$ are not connected. Therefore, the cardinality of a maximal clique of $M_{U,B} = (V, E)$ is not greater than n .

If the cardinality of a maximal clique $S \subseteq V$ is n , then, for every i , $|S \cap (\{i\} \times \rho_i)| = 1$. Then, denoting the only element of $S \cap (\{i\} \times \rho_i)$ by v_i , we see that the assignment $f(x_i) = v_i$ satisfies all constraints. The opposite is also true, i.e., if the constraints $\langle \rho_i \rangle_{1 \leq i \leq n}, \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ can be satisfied by some assignment f , then $\{(i, f(x_i)) \mid 1 \leq i \leq n\}$ is a clique of cardinality n . ■

Hence, $2 - \text{MinHom}(\Gamma)$ can be reduced to finding a maximal-size clique $S \subseteq V$ of a microstructure graph that minimizes the following value:

$$\sum_{(i,a) \in S} w_{ia}.$$

Definition 6.5. Let *MMClique* (Minimal weight among maximal-size cliques) denote the following minimization problem:

Instance: A graph $G = (V, E)$ and weights $w_i \in \mathbb{N}, i \in V$.

Solution: A maximal-size clique $K \subseteq V$ of G .

Measure: $\sum_{v \in K} w_v$.

The following theorem connects perfect microstructure graphs and the complexity of *MinHom*.

Theorem 6.6. Suppose we are given a class of conservative functions F containing a majority operation. If the microstructure graph is perfect for arbitrary arc-consistent and path-consistent instances of $2 - \text{MinHom}(\text{Inv}(F))$, then F is tractable.

Proof. Recall that a graph $G = (V, E)$ is called perfect if for every induced subgraph the chromatic number is equal to the clique number.

For a graph $G = (V, E)$, the following polytope is called the *fractional stable set polytope*:

$$\begin{cases} \sum_{v \in K} x_v \leq 1, \text{ where } K \text{ is a clique in } G \\ x_v \geq 0, v \in V \end{cases}$$

By a well-known theorem of Lovasz[10], a graph $G = (V, E)$ is perfect if and only if its fractional stable set polytope equals the convex hull of the characteristic vectors of independent sets in G . By the vertex packing problem we mean the weighted version

of maximum independent set. It is easy to see that vertex packing in perfect graphs is equivalent to optimizing a linear function over the fractional stable set polytope. There is a polynomial algorithm for solving the vertex packing in perfect graphs[11]. Using well-known results[10, 20] about polynomial equivalence between the separation and optimization of linear function on polytopes we obtain that there is a polynomial algorithm that takes a perfect graph $G = (V, E)$, a rational vector $a_v, v \in V$ as input, and checks whether the vector is in the fractional stable set polytope of G or not. If not, it finds a hyperplane (given by rational vectors) that separates $a_v, v \in V$ from the polytope.

Therefore, there exists a polynomial separation algorithm for the fractional stable set polytope of a perfect graph with addition of the following equality: $\sum_{v \in V} x_v = \alpha(G)$ where $\alpha(G)$ is independence number of the given graph G . That is, we have a polynomial algorithm for the following task:

$$\left\{ \begin{array}{l} \sum_{v \in K} x_v \leq 1, \text{ where } K \text{ is a clique in } G \\ x_v \geq 0, v \in V \\ \sum_{v \in V} x_v = \alpha(G) \\ \sum_{v \in V} w_v x_v \rightarrow \min \end{array} \right.$$

It is easy to see that this task coincides with MMClique for the complement of G . Since the complement of a perfect graph is perfect, MMClique for perfect graphs is polynomial-time solvable, too. \blacksquare

Definition 6.7. A cycle C_{2k+1} , $k \geq 2$, is called an *odd hole* and its complement graph an *odd antihole*.

In Section 8 we will use the following conjecture of Berge, which was proved in [6].

Theorem 6.8. A graph is perfect if and only if it does not contain an induced subgraph isomorphic to an odd hole or antihole.

We say that a graph is of type S_{2k+1} , $k \geq 2$ if it is isomorphic to the graph with vertex set $\{0, 1, \dots, 2k\}$, where vertices $i \pmod{2k+1}$, $i+1 \pmod{2k+1}$ are not connected and vertices $i \pmod{2k+1}$, $i+2 \pmod{2k+1}$ are connected. Other pairs can be connected arbitrarily. Obviously, every odd hole or antihole is of one of types S_{2k+1} , $k \geq 2$.

7. Arithmetical deadlocks

The key idea for the proof of the polynomial case of Theorem 3.7 is to show that path- and arc-consistent instances of $2 - \text{MinHom}(\text{Inv}(F))$ have a perfect microstructure graph. We will prove this by showing that the microstructure graph forbids certain types of subgraphs. The exact formulation of the result can be found below in Theorem 8.1. This theorem uses the nonexistence of structures called *arithmetical deadlocks* which are introduced in this section.

Definition 7.1. Suppose H is a conservative set of functions over D , $m \in H$ is an arithmetical operation on $B \subseteq \{\{x, y\} \mid x, y \in D, x \neq y\}$ and a pair $\phi, \psi \in H$ is a tournament pair on \overline{B} . An instance of $2 - \text{MinHom}(\text{Inv}(H))$ with constraints pair $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ is called an *odd arithmetical deadlock* if there is a subset $\{i_0, \dots, i_{k-1}\} \subseteq \{1, \dots, n\}$, $k \geq 3$ of odd cardinality and $\{x_0, y_0\}, \dots, \{x_{k-1}, y_{k-1}\} \in B$, such that for

$0 \leq s \leq k-1$: $\rho_{i_s, i_{s \oplus 1}} \cap \{x_s, y_s\} \times \{x_{s \oplus 1}, y_{s \oplus 1}\} = \begin{smallmatrix} x_s & \times & x_{s \oplus 1} \\ y_s & & y_{s \oplus 1} \end{smallmatrix}$, where $i \oplus j$ denotes $i+j \pmod k$. The subset $\{i_0, \dots, i_{k-1}\}$ is called a *deadlock subset*.

Theorem 7.2. *Suppose H is a conservative set of functions over D , $m \in H$ is an arithmetical operation on $B \subseteq \{\{x, y\} \mid x, y \in D, x \neq y\}$ and a pair $\phi, \psi \in H$ is a tournament pair on \overline{B} . If an instance of $2 - \text{MinHom}(\text{Inv}(H))$ is arc- and path-consistent, then it cannot be an odd arithmetical deadlock.*

We will begin by introducing some technical concepts from the theory of *CSP* which we will need in the proof of Theorem 7.2. An algebra \mathbb{A} is said to be of *type* \mathfrak{F} if its operations are indexed by elements of the set \mathfrak{F} , called terms. For every $f \in \mathfrak{F}$, the corresponding operation is denoted by $f^{\mathbb{A}}$. The universe of an algebra \mathbb{A}_i is denoted by A_i . Recall that $\rho^t = \{(y, x) \mid (x, y) \in \rho\}$.

Definition 7.3. Let a finite set of indexes I be given and every index $i \in I$ corresponds to some algebra \mathbb{A}_i of type \mathfrak{F} . A set of indexed multi-domain predicates over $\{\mathbb{A}_i\}_{i \in I}$ is a pair $\langle \rho_i \rangle_{i \in I}, \langle \rho_{kl} \rangle_{k \neq l \in I}$, where for each i and $k \neq l$, ρ_i is a subalgebra of \mathbb{A}_i and ρ_{kl} is a subalgebra of $\mathbb{A}_k \times \mathbb{A}_l$. We assume that $\rho_{kl} = \rho_{lk}^t$.

Definition 7.4. A set of indexed multi-domain predicates $\langle \rho_i \rangle_{i \in I}, \langle \rho_{kl} \rangle_{k \neq l \in I}$ over $\{\mathbb{A}_i\}_{i \in I}$ is called *arc-consistent* if for distinct $i, j \in I$: $\text{Pr}_1 \rho_{ij} = \rho_i, \text{Pr}_2 \rho_{ij} = \rho_j$.

Definition 7.5. A set of indexed multi-domain predicates $\langle \rho_i \rangle_{i \in I}, \langle \rho_{kl} \rangle_{k \neq l \in I}$ over $\{\mathbb{A}_i\}_{i \in I}$ is called *path-consistent* if for any distinct $i, j, k \in I$: $\rho_{ik} \subseteq \rho_{ij} \circ \rho_{jk}$.

Introduce the notation $P_i = \{\{x, y\} \mid x, y \in A_i, x \neq y\}$.

Definition 7.6. Assume that algebras $\{\mathbb{A}_i\}_{i \in I}$ are of type \mathfrak{F} , that they are conservative, and $B_i \subseteq P_i, i \in I$. A term $m \in \mathfrak{F}$ is called *arithmetical on* $\{B_i\}_{i \in I}$, if for any $i \in I$ m^{A_i} is arithmetical on B_i . A pair of terms $\phi, \psi \in \mathfrak{F}$ is called a *tournament pair on* $\{B_i\}_{i \in I}$, if for any $i \in I$ a pair ϕ^{A_i}, ψ^{A_i} is a tournament pair on B_i .

We now generalize the notion of an *odd arithmetical deadlock* to multi-domain constraints.

Definition 7.7. Assume that algebras $\{\mathbb{A}_i\}_{i \in I}$ are of type \mathfrak{F} , that they are conservative, and $B_i \subseteq P_i, i \in I$. Furthermore, assume $m \in \mathfrak{F}$ is an arithmetical term on $\{B_i\}_{i \in I}$ and a pair $\phi, \psi \in \mathfrak{F}$ is a tournament pair on $\{P_i/B_i\}_{i \in I}$. Then, the set of indexed multi-domain predicates $\langle \rho_i \rangle_{i \in I}, \langle \rho_{kl} \rangle_{k \neq l \in I}$ over $\{\mathbb{A}_i\}_{i \in I}$ is called an *odd arithmetical deadlock* if there is a subset $\{i_0, \dots, i_{n-1}\} \subseteq I, n \geq 3$ of odd cardinality and $\{x_0, y_0\} \in B_{i_0}, \dots, \{x_{n-1}, y_{n-1}\} \in B_{i_{n-1}}$, such that for $0 \leq k \leq n-1$: $\rho_{i_k, i_{k \oplus 1}} \cap \{x_k, y_k\} \times \{x_{k \oplus 1}, y_{k \oplus 1}\} = \begin{smallmatrix} x_k & \times & x_{k \oplus 1} \\ y_k & & y_{k \oplus 1} \end{smallmatrix}$, where $i \oplus j$ denotes $i+j \pmod n$. The subset $\{i_0, \dots, i_{n-1}\}$ is called a *deadlock subset*.

We will now prove the following theorem, which is a generalization of Theorem 7.2.

Theorem 7.8. *Suppose $m \in \mathfrak{F}$ is an arithmetical term on $\{B_i\}_{i \in I}$, and a pair $\phi, \psi \in \mathfrak{F}$ is a tournament pair on $\{P_i/B_i\}_{i \in I}$. If a set of indexed multi-domain predicates $\langle \rho_i \rangle_{i \in I}, \langle \rho_{kl} \rangle_{k \neq l \in I}$ over $\{\mathbb{A}_i\}_{i \in I}$ is arc- and path-consistent, then it cannot be an odd arithmetical deadlock.*

Any instance of $2 - \text{MinHom}(\text{Inv}(H))$ can be considered as a set of indexed multi-domain predicates over $\{\mathbb{A}_i\}_{i \in I}$ where I is a set of variables and $\mathbb{A}_i = \mathbb{A}$. By defining $B_i = B$ we see that Theorem 7.2 is a special case of Theorem 7.8. Before proving Theorem 7.8, we need to prove some preliminary lemmas.

Recall that a congruence of an algebra \mathbb{A} is an equivalence relation on A that is a subalgebra of \mathbb{A}^2 . If θ is a congruence of \mathbb{A} and $a \in A$, then equivalence class of θ containing a is denoted by a^θ . If for each $s \in I$, θ_s is a congruence of \mathbb{A}_s , then $\rho_i/\theta_i = \{x^{\theta_i} \mid x \in \rho_i\}$ and $\rho_{kl}/(\theta_k \times \theta_l) = \{(x^{\theta_k}, y^{\theta_l}) \mid (x, y) \in \rho_{kl}\}$, which we view as subalgebras of \mathbb{A}_i/θ_i and $(\mathbb{A}_k/\theta_k) \times (\mathbb{A}_l/\theta_l)$.

Lemma 7.9. *Let θ_i be a congruence of \mathbb{A}_i for each $i \in I$ and assume that a set of indexed multi-domain predicates $\langle \rho_i \rangle_{i \in I}, \langle \rho_{kl} \rangle_{k \neq l \in I}$ over $\{\mathbb{A}_i\}_{i \in I}$ is arc- and path-consistent. Then a set of indexed multi-domain predicates $\{\rho_i/\theta_i\}_{i \in I}, \{\rho_{kl}/(\theta_k \times \theta_l)\}_{k \neq l \in I}$ over $\{\mathbb{A}_i/\theta_i\}_{i \in I}$ is arc- and path-consistent, too.*

Proof. Let $n_i : \mathbb{A}_i \rightarrow \mathbb{A}_i/\theta_i$ be natural homomorphisms, i.e., $n_i(x) = x^{\theta_i}$. Obviously, $\rho_i/\theta_i = \{n_i(x) \mid x \in \rho_i\}$, $\rho_{kl}/(\theta_k \times \theta_l) = \{(n_k(x), n_l(y)) \mid (x, y) \in \rho_{kl}\}$ and $\text{Pr}_1[\rho_{kl}/(\theta_k \times \theta_l)] = \{n_k(x) \mid x \in \text{Pr}_1 \rho_{kl}\} = \text{Pr}_1 \rho_{kl}/\theta_k$. Analogously, we can prove that $\text{Pr}_2[\rho_{kl}/(\theta_k \times \theta_l)] = \text{Pr}_2 \rho_{kl}/\theta_l$.

From arc-consistency it follows that $\text{Pr}_1 \rho_{kl} = \rho_k, \text{Pr}_2 \rho_{kl} = \rho_l$, and we have $\text{Pr}_1[\rho_{kl}/(\theta_k \times \theta_l)] = \rho_k/\theta_k, \text{Pr}_2[\rho_{kl}/(\theta_k \times \theta_l)] = \rho_l/\theta_l$. This is equivalent to arc-consistency of the set $\{\rho_i/\theta_i\}_{i \in I}, \{\rho_{kl}/(\theta_k \times \theta_l)\}_{k \neq l \in I}$.

The path-consistency condition $\rho_{ik} \subseteq \rho_{ij} \circ \rho_{jk}$ gives us:

$$\begin{aligned} & \rho_{ij}/(\theta_i \times \theta_j) \circ \rho_{jk}/(\theta_j \times \theta_k) = \\ & = \{(n_i(x), n_j(y)) \mid (x, y) \in \rho_{ij}\} \circ \{(n_j(z), n_k(t)) \mid (z, t) \in \rho_{jk}\} \supseteq \\ & \supseteq \{(n_i(x), n_k(t)) \mid (x, y) \in \rho_{ij}, (y, t) \in \rho_{jk}\} \supseteq \\ & \supseteq \{(n_i(x), n_k(t)) \mid (x, t) \in \rho_{ik}\} = \rho_{ik}/(\theta_i \times \theta_k) \end{aligned}$$

This is equivalent to path-consistency of $\{\rho_i/\theta_i\}_{i \in I}$ and $\{\rho_{kl}/(\theta_k \times \theta_l)\}_{k \neq l \in I}$. \blacksquare

For $\rho \subseteq A_1 \times A_2$, let $\rho(x, \cdot) = \{y \mid \rho(x, y)\}$ and $\rho(\cdot, x) = \{y \mid \rho(y, x)\}$.

Lemma 7.10. *Suppose algebras $\{\mathbb{A}_i\}_{i=1,2}$ of type \mathfrak{F} are conservative and $B_i \subseteq P_i, i = 1, 2$. Furthermore, assume that $m \in \mathfrak{F}$ is an arithmetical term on $B_i, i = 1, 2$, and a pair $\phi, \psi \in \mathfrak{F}$ is a tournament pair on $P_i/B_i, i = 1, 2$. If ρ is a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ and there are $\{x_i, y_i\} \in B_i, i = 1, 2$, such that $\rho \cap \{x_1, y_1\} \times \{x_2, y_2\} = \begin{smallmatrix} x_1 & \times & x_2 \\ y_1 & & y_2 \end{smallmatrix}$, then $\rho(x_1, \cdot) \cap \rho(y_1, \cdot) = \emptyset$ and $\rho(\cdot, x_2) \cap \rho(\cdot, y_2) = \emptyset$.*

Proof. Suppose, for example, that $t \in \rho(x_1, \cdot) \cap \rho(y_1, \cdot)$. Then, if $\{x_2, t\} \in B_2$, we have:

$$\begin{pmatrix} x_1 \\ t \end{pmatrix}, \begin{pmatrix} y_1 \\ t \end{pmatrix}, \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} \in \rho \Rightarrow \begin{pmatrix} m^{\mathbb{A}_1}(x_1, y_1, y_1) \\ m^{\mathbb{A}_2}(t, t, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \rho$$

If $\{x_2, t\} \in P_2/B_2$, then there is a $\lambda \in \mathfrak{F} : \downarrow_{x_2}^t \lambda^{\mathbb{A}_2}$ where either $\lambda = \phi$ or $\lambda = \psi$ and we

have:

$$\begin{pmatrix} x_1 \\ t \end{pmatrix}, \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} \in \rho \Rightarrow \begin{pmatrix} \lambda^{\mathbb{A}_1}(x_1, y_1) \\ \lambda^{\mathbb{A}_2}(t, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \rho$$

Now we see that $\rho(x_1, \cdot) \cap \rho(y_1, \cdot) = \emptyset$ (analogously $\rho(\cdot, x_2) \cap \rho(\cdot, y_2) = \emptyset$). \blacksquare

For $\rho \subseteq A_1 \times A_2$, θ_1^ρ and θ_2^ρ denote the transitive closures of $\rho \circ \rho^t$ and $\rho^t \circ \rho$ respectively.

Lemma 7.11. *Suppose algebras $\{\mathbb{A}_i\}_{i=1,2}$ of type \mathfrak{F} are conservative and $B_i \subseteq P_i, i = 1, 2$. Suppose also that $m \in \mathfrak{F}$ is arithmetical term on $B_i, i = 1, 2$, and a pair $\phi, \psi \in \mathfrak{F}$ is a tournament pair on $P_i/B_i, i = 1, 2$. If ρ is a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ and there are $\{x_i, y_i\} \in B_i, i = 1, 2$, such that $\rho \cap \{x_1, y_1\} \times \{x_2, y_2\} = \begin{smallmatrix} x_1 & \times & x_2 \\ y_1 & & y_2 \end{smallmatrix}$, then $x_i^{\theta_i^\rho} \neq y_i^{\theta_i^\rho}, i = 1, 2$.*

Proof. Note that for $x \in A_1$, the equivalence class $x^{\theta_1^\rho}$ can be obtained by the following procedure: $U_1 = \{x\}$, $U_2 = \{t | \exists y \in U_1 \rho(y, t)\}$, $U_3 = \{t | \exists y \in U_2 \rho(t, y)\}$, $U_4 = \{t | \exists y \in U_3 \rho(y, t)\}$ and so on. The resulting equivalence class is $U_1 \cup U_3 \cup U_5 \dots$. Consider this process for elements x_1, y_1 and denote the corresponding sets by $U_1^{x_1}, U_2^{x_1}, \dots$ and $U_1^{y_1}, U_2^{y_1}, \dots$. We prove by induction that $U_s^{x_1} \cap U_s^{y_1} = \emptyset$ and $\delta_k \triangleq (U_k^{x_1})^2 \cup (U_k^{y_1})^2$ is a congruence of $\mathbb{A}_1|_{U_k^{x_1} \cup U_k^{y_1}}$, if k is odd, or of $\mathbb{A}_2|_{U_k^{x_1} \cup U_k^{y_1}}$, if k is even.

Base of induction. Obviously, $U_1^{x_1} \cap U_1^{y_1} = \emptyset$. Since $\rho' = \rho \cap \{x_1, y_1\} \times \{x_2, y_2\} = \begin{smallmatrix} x_1 & \times & x_2 \\ y_1 & & y_2 \end{smallmatrix}$ is a subalgebra of $\mathbb{A}_1|_{\{x_1, y_1\}} \times \mathbb{A}_2|_{\{x_2, y_2\}}$, we see that $(U_1^{x_1})^2 \cup (U_1^{y_1})^2 = \theta_1^{\rho'}$ is a congruence of $\mathbb{A}_1|_{\{x_1, y_1\}}$.

Suppose the assertion is true for $s \leq k$. Consider the case when k is even (the odd case is analogous). Let $\rho' = \rho \cap (A_1 \times (U_k^{x_1} \cup U_k^{y_1}))$. Clearly, $\rho' / (=^{A_1} \times \delta_k)$ is a subalgebra of $\mathbb{A}_1 \times (\mathbb{A}_2|_{U_k^{x_1} \cup U_k^{y_1}} / \delta_k)$ and from $y_2 \in U_k^{x_1}, x_2 \in U_k^{y_1}$ we have

$$\begin{aligned} U_{k+1}^{x_1} &= \rho' / (=^{A_1} \times \delta_k) \left(\cdot, y_2^{\delta_k} \right) \\ U_{k+1}^{y_1} &= \rho' / (=^{A_1} \times \delta_k) \left(\cdot, x_2^{\delta_k} \right) \end{aligned}$$

A pair of algebras $\mathbb{A}_1, \mathbb{A}_2|_{U_k^{x_1} \cup U_k^{y_1}} / \delta_k$ of type \mathfrak{F} obviously satisfy conditions of Lemma 7.10. Since $\rho(x_1, \cdot) \subseteq U_k^{x_1}$ and $\rho(y_1, \cdot) \subseteq U_k^{y_1}$, we have

$$\rho' / (=^{A_1} \times \delta_k) \cap \{x_1, y_1\} \times \{x_2^{\delta_k}, y_2^{\delta_k}\} = \begin{smallmatrix} x_1 & \times & x_2^{\delta_k} \\ y_1 & & y_2^{\delta_k} \end{smallmatrix}.$$

From Lemma 7.10 we see that

$$\rho' / (=^{A_1} \times \delta_k) \left(\cdot, y_2^{\delta_k} \right) \cap \rho' / (=^{A_1} \times \delta_k) \left(\cdot, x_2^{\delta_k} \right) = \emptyset$$

which is equivalent to $U_{k+1}^{x_1} \cap U_{k+1}^{y_1} = \emptyset$.

From the emptiness of this intersection, we conclude that the predicate $\sigma = \theta_1^{\rho' / (=^{A_1} \times \delta_k)}$ is a congruence and equals to $(U_{k+1}^{x_1})^2 \cup (U_{k+1}^{y_1})^2$, and the induction is completed. \blacksquare

Lemma 7.12. *Suppose \mathbb{A} is three-element algebra containing an operation $h : A^3 \rightarrow A$ that is arithmetical on $\{\{a, b\} | a, b \in A, a \neq b\}$. Then, there cannot be two different nontrivial (i.e. not equal to A^2 or $=^A$) congruences of this algebra.*

Proof. We give a proof by contradiction. Without loss of generality we can assume that $A = \{0, 1, 2\}$ and $\sim^1 = \{(0, 0), (1, 1), (2, 2), (0, 1)\}$, $\sim^2 = \{(0, 0), (1, 1), (2, 2), (1, 2)\}$. Since h preserve \sim^1 , we have:

$$\begin{aligned} h(1, 1, 2) &= 2 \\ h(0, 1, 2) &=? \Rightarrow h(0, 1, 2) = 2. \end{aligned}$$

Preservation of \sim^2 leads to contradiction:

$$\begin{aligned} h(0, 1, 1) &= 0 \\ h(0, 1, 2) &=? \Rightarrow h(0, 1, 2) = 0. \end{aligned}$$

\blacksquare

Proof of Theorem 7.8. Suppose to the contrary that there exists a set of indexed multi-domain predicates that is an odd arithmetical deadlock. We can assume that $I = \{0, \dots, 2d\}$ and $\{x_0, y_0\} \in B_0, \dots, \{x_{2d}, y_{2d}\} \in B_{2d}$, such that $\rho_{k,k\oplus 1} \cap \{x_k, y_k\} \times \{x_{k\oplus 1}, y_{k\oplus 1}\} = \begin{smallmatrix} x_k & \times & x_{k\oplus 1} \\ y_k & & y_{k\oplus 1} \end{smallmatrix}$, where $i \oplus j$ denotes $i + j \pmod{2d+1}$.

Consider the predicates $\rho_{k\oplus 1, k}$ and $\rho_{k, k\oplus 1}$. Let $\theta-$ and $\theta+$ denote congruences $\theta_2^{\rho_{k\oplus 1, k}}, \theta_1^{\rho_{k, k\oplus 1}}$ consistently. By Lemma 7.11, $x_k^{\theta+} \neq y_k^{\theta+}$. Obviously, $\rho_{k, k\oplus 1}(\cdot, x_{k\oplus 1}) \subseteq y_k^{\theta+}$ and $\rho_{k, k\oplus 1}(\cdot, y_{k\oplus 1}) \subseteq x_k^{\theta+}$. Therefore, we conclude that

$$\rho_{k, k\oplus 1} / ((\theta+) \times (=^{A_{k\oplus 1}})) \cap \{x_k^{\theta+}, y_k^{\theta+}\} \times \{x_{k\oplus 1}, y_{k\oplus 1}\} = \begin{smallmatrix} x_k^{\theta+} & \times & x_{k\oplus 1} \\ y_k^{\theta+} & & y_{k\oplus 1} \end{smallmatrix}.$$

Let us show that $\rho_{k\oplus 1, k}(x_{k\oplus 1}, \cdot) \cap x_k^{\theta+} = \emptyset$ and $\rho_{k\oplus 1, k}(y_{k\oplus 1}, \cdot) \cap y_k^{\theta+} = \emptyset$. Suppose to the contrary that the first one is false (the other case is absolutely analogous), i.e. $t \in \rho_{k\oplus 1, k}(x_{k\oplus 1}, \cdot) \cap x_k^{\theta+}$. From $\rho_{k\oplus 1, k}(x_{k\oplus 1}, t)$, we see that $(t, y_k) \in \theta-$. But, from $t \in x_k^{\theta+}$, we conclude that $(t, x_k) \in \theta+$. Consider the three-element algebra $\mathbb{A}_k|_{\{x_k, y_k, t\}}$. The congruences $\theta+$, $\theta-$ restricted to that algebra are equal to $\{\{x_k, t\}, \{y_k\}\}$ and $\{\{y_k, t\}, \{x_k\}\}$, since, by Lemma 7.11, $x_k^{\theta+} \neq y_k^{\theta+}$ and $x_k^{\theta-} \neq y_k^{\theta-}$. It is easy to see that the three-element conservative algebra $\mathbb{A}_k|_{\{x_k, y_k, t\}}$ with $\{x_k, y_k\} \in B_k$ has such congruences only if m is arithmetical on $\{\{x_k, y_k\}, \{y_k, t\}, \{x_k, t\}\}$. This contradicts Lemma 7.12.

From $\rho_{k\oplus 1, k}(x_{k\oplus 1}, \cdot) \cap x_k^{\theta+} = \emptyset$ and $\rho_{k\oplus 1, k}(y_{k\oplus 1}, \cdot) \cap y_k^{\theta+} = \emptyset$, we conclude that

$$\rho_{k\oplus 1, k} / ((=^{A_{k\oplus 1}}) \times (\theta+)) \cap \{x_{k\oplus 1}, y_{k\oplus 1}\} \times \{x_k^{\theta+}, y_k^{\theta+}\} = \begin{smallmatrix} x_{k\oplus 1} & \times & x_k^{\theta+} \\ y_{k\oplus 1} & & y_k^{\theta+} \end{smallmatrix}.$$

Therefore, changing a system of one-type algebras $\{\mathbb{A}_i\}_{i \in I}$ to $\{\mathbb{A}_i/\lambda_i\}_{i \in I}$ where

$$\lambda_i = \begin{cases} \theta_1^{\rho_{k, k\oplus 1}}, & \text{if } i = k \\ =^{A_i}, & \text{otherwise} \end{cases}$$

we obtain, by Lemma 7.9, an arc- and path-consistent set of indexed predicates $\{\rho_i/\lambda_i\}_{i \in I}, \{\rho_{kl}/(\lambda_k \times \lambda_l)\}_{k \neq l \in I}$. The resulting set of predicates will be an odd arithmetical deadlock, too.

Analogously, we can prove that changing a system of one-type algebras $\{\mathbb{A}_i\}_{i \in I}$ to $\{\mathbb{A}_i/\lambda_i\}_{i \in I}$, where

$$\lambda_i = \begin{cases} \theta_2^{\rho_{k\oplus 1, k}}, & \text{if } i = k \\ =^{A_i}, & \text{otherwise} \end{cases}$$

result in an arc- and path-consistent set of indexed predicates $\{\rho_i/\lambda_i\}_{i \in I}, \{\rho_{kl}/(\lambda_k \times \lambda_l)\}_{k \neq l \in I}$, which will be an odd arithmetical deadlock.

By using those transformations for different k successively, we eventually obtain an arc- and path-consistent $\{\rho'_i\}_{i \in I}, \{\rho'_{kl}\}_{k \neq l \in I}$, such that $\forall k \rho'_{k, k\oplus 1} \cap \{x'_k, y'_k\} \times \{x'_{k\oplus 1}, y'_{k\oplus 1}\} = \begin{smallmatrix} x'_k & \times & x'_{k\oplus 1} \\ y'_k & & y'_{k\oplus 1} \end{smallmatrix}$ and $\forall k \rho_{k, k\oplus 1}(\cdot, x'_{k\oplus 1}) = \{y'_k\}, \rho_{k, k\oplus 1}(\cdot, y'_{k\oplus 1}) = \{x'_k\}, \rho_{k\oplus 1, k}(x'_{k\oplus 1}, \cdot) = \{y'_k\}$ and $\rho_{k\oplus 1, k}(y'_{k\oplus 1}, \cdot) = \{x'_k\}$. We show that there is no such set.

From path-consistency we conclude that for any $0 \leq k < l \leq 2d$: $\rho'_{kl} \subseteq \rho'_{k, k+1} \circ \rho'_{k+1, k+2} \circ \dots \circ \rho'_{l-1, l}$. Hence,

$$\rho'_{k, k+1} \circ \rho'_{k+1, k+2} \circ \dots \circ \rho'_{l-1, l}(x'_k, \cdot) = \begin{cases} \{x'_l\}, & \text{if } l - k \text{ even} \\ \{y'_l\}, & \text{if } l - k \text{ odd} \end{cases}$$

Since $\rho'_{kl}(x'_k, \cdot)$ is not empty, we see that

$$\rho'_{kl}(x'_k, \cdot) = \begin{cases} \{x'_l\}, & \text{if } l - k \text{ even} \\ \{y'_l\}, & \text{if } l - k \text{ odd} \end{cases}$$

However, we have $\rho'_{0,2d} \cap \{x'_0, y'_0\} \times \{x'_{2d}, y'_{2d}\} = \begin{matrix} x'_0 & \times & x'_{2d} \\ y'_0 & \times & y'_{2d} \end{matrix}$ which contradicts that $\rho'_{0,2d}(x'_0, \cdot) = \{x'_{2d}\}$. \blacksquare

8. Final step in a proof of polynomial case

Theorem 8.1. *Suppose that F satisfies the necessary local conditions and that the graph $T_F = (M^o, P)$ is bipartite. Then for every path- and arc-consistent instance of $2 - \text{MinHom}(\text{Inv}(F))$, its microstructure graph forbids subgraphs of type $S_{2p+1}, p \geq 2$.*

Proof. Suppose to the contrary that we have a arc- and path-consistent instance $I = (X, U, B, w)$ of $2 - \text{MinHom}(\text{Inv}(F))$ with constraints pair $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ and its microstructure graph has a subgraph of type $S_{2p+1}, p \geq 2$. For convenience, let us introduce $\rho_{ii} = \{(a, a) \mid a \in \rho_i\}$. Then, there is a set of pairs $\{(i_0, b_0), (i_1, b_1), \dots, (i_{2p}, b_{2p})\}$, such that for $0 \leq l \leq 2p$: $(b_l, b_{l \oplus 1}) \notin \rho_{i_l i_{l \oplus 1}}$ and $(b_l, b_{l \oplus 2}) \in \rho_{i_l i_{l \oplus 2}}$, where $i \oplus j$ denotes $i + j \pmod{2p+1}$.

From $(b_l, b_{l \oplus 2}) \in \rho_{i_l i_{l \oplus 2}}$ and the path-consistency condition $\rho_{i_l i_{l \oplus 2}} \subseteq \rho_{i_l i_{l \oplus 1}} \circ \rho_{i_{l \oplus 1} i_{l \oplus 2}}$, we see that there is $a_{l \oplus 1}$, such that $(b_l, a_{l \oplus 1}) \in \rho_{i_l i_{l \oplus 1}}$ and $(a_{l \oplus 1}, b_{l \oplus 2}) \in \rho_{i_{l \oplus 1} i_{l \oplus 2}}$.

Consider the predicate $\rho'_{l, l \oplus 1} = \rho_{i_l i_{l \oplus 1}} \cap \{a_l, b_l\} \times \{a_{l \oplus 1}, b_{l \oplus 1}\} \in \text{Inv}(F)$. Obviously, $\rho'_{l, l \oplus 1}$ equals to either $\begin{matrix} a_l & \times & a_{l \oplus 1} \\ b_l & \times & b_{l \oplus 1} \end{matrix}$ or $\begin{matrix} a_l & \times & a_{l \oplus 1} \\ b_l & \times & b_{l \oplus 1} \end{matrix}$.

Let us show that if $\{a_l, b_l\} \in \overline{M}$, then $\{a_{l \oplus 1}, b_{l \oplus 1}\} \in \overline{M}$, too. Assume to the contrary that $\{a_{l \oplus 1}, b_{l \oplus 1}\} \in M$. Then, by Theorem 5.3, there is a $\phi \in F : \begin{matrix} a_{l \oplus 1} \\ \downarrow \phi \\ b_{l \oplus 1} \end{matrix}$, where $\phi|_{\{a_l, b_l\}}$ is a projection on the first coordinate. In this case, ϕ preserves neither $\begin{matrix} a_l & \times & a_{l \oplus 1} \\ b_l & \times & b_{l \oplus 1} \end{matrix}$ nor $\begin{matrix} a_l & \times & a_{l \oplus 1} \\ b_l & \times & b_{l \oplus 1} \end{matrix}$, because

$$\begin{pmatrix} b_l \\ b_{l \oplus 1} \end{pmatrix} = \begin{pmatrix} \phi(b_l, a_l) \\ \phi(a_{l \oplus 1}, b_{l \oplus 1}) \end{pmatrix}.$$

Hence, we need to consider two cases only: 1) $\forall l \{a_l, b_l\} \in M$ and 2) $\forall l \{a_l, b_l\} \in \overline{M}$. In the first case, we have $\langle (a_l, b_l), (a_{l \oplus 1}, b_{l \oplus 1}) \rangle \in P$, i.e., there is an odd cycle in T_F which contradicts that T_F is bipartite.

Now, consider the case $\forall l \{a_l, b_l\} \in \overline{M}$. By Theorem 5.4, there is a function $m \in F$, arithmetical on \overline{M} . If $\rho'_{l, l \oplus 1} = \begin{matrix} a_l & \times & a_{l \oplus 1} \\ b_l & \times & b_{l \oplus 1} \end{matrix}$, then we have that

$$\begin{pmatrix} b_l \\ b_{l \oplus 1} \end{pmatrix} = \begin{pmatrix} m(a_l, a_l, b_l) \\ m(b_{l \oplus 1}, a_{l \oplus 1}, a_{l \oplus 1}) \end{pmatrix} \in \rho'_{l, l \oplus 1}$$

and $\rho'_{l, l \oplus 1} = \begin{matrix} a_l & \times & a_{l \oplus 1} \\ b_l & \times & b_{l \oplus 1} \end{matrix}$.

Consider the set $\{i_0, i_1, \dots, i_{2p}\}$. Suppose first that all i_0, i_1, \dots, i_{2p} are distinct. Then, Theorems 5.3 and 5.4 show us that we have an arithmetical operation $m \in F$ on \overline{M} and a tournament pair $\phi, \psi \in F$ on M . It is easy to see that an instance of $2 - \text{MinHom}(\text{Inv}(F))$ with constraints pair $U = \langle \rho_i \rangle_{1 \leq i \leq n}$, $B = \langle \rho_{kl} \rangle_{1 \leq k \neq l \leq n}$ is an odd arithmetical deadlock where $\{i_0, i_1, \dots, i_{2p}\}$ is a deadlock set. This contradicts that I is arc- and path-consistent.

The case when the elements i_0, i_1, \dots, i_{2p} are not distinct can be reduced to the previous case by the following trick: introduce a new set of variables $X' =$

$\{(i_0, 0), (i_1, 1), \dots, (i_{2p}, 2p)\}$ and $\rho_{(i_s, s)} = \rho_{i_s}$, where $0 \leq s \leq 2p$. If $i_m \neq i_n$, then $\rho_{(i_m, m), (i_n, n)} = \rho_{i_m, i_n}$, else $\rho_{(i_m, m), (i_n, n)} = \{(a, a) | a \in \rho_{i_m}\}$. It is easy to see that an instance with constraints pair $U = \{\rho_i\}_{i \in X'}, B = \{\rho_{kl}\}_{k \neq l \in X'}$ satisfy the conditions of Theorem 7.2 and is an odd arithmetical deadlock, where the set $\{(i_0, 0), (i_1, 1), \dots, (i_{2p}, 2p)\}$ is a deadlock set. Therefore, we have a contradiction. ■

Proof of polynomial case of Theorem 3.7. The conditions of Theorem 3.7 coincides with the conditions of Theorem 8.1 so the microstructure graph of an arc- and path-consistent instance forbids subgraphs of type $S_{2p+1}, p \geq 2$. By Theorem 6.8, it is perfect and, by Theorem 6.6, we see that the class F is tractable. ■

Theorems 3.6 and 3.7 give the required dichotomy for conservative algebras, which implies the dichotomy for conservative constraint languages. By Theorem 2.9, we have the following general dichotomy.

Theorem 8.2. *If $\text{MinHom}(\Gamma)$ is not tractable then it is NP-hard.*

9. Tractable constraint languages

It is possible to reformulate our results in terms of constraint languages. Let lin_{a_0, a_1} denote the predicate $\{(a_x, a_y, a_z) | x, y, z \in \{0, 1\}, x \oplus y \oplus z = 0\}$ where \oplus denotes an addition modulo 2. For example, $\text{lin}_{0,1} = \{(x, y, z) | x, y, z \in \{0, 1\}, x \oplus y \oplus z = 0\}$.

Theorem 9.1. *Suppose Γ is a constraint language over A which is a conservative relational clone, then either*

- $\exists a \neq b \in A$ such that $\begin{smallmatrix} a \\ \diagdown \\ b \end{smallmatrix} \in \Gamma$, or
- $\exists a \neq b \in A$ such that $\text{lin}_{a,b} \in \Gamma$, or
- $\exists a_0 \neq b_0, \dots, a_{2k} \neq b_{2k} \in A$ such that $\begin{smallmatrix} a_0 & \diagdown & a_1 \\ b_0 & \diagdown & b_1 \end{smallmatrix}, \dots, \begin{smallmatrix} a_{2k-1} & \diagdown & a_{2k} \\ b_{2k-1} & \diagdown & b_{2k} \end{smallmatrix}, \begin{smallmatrix} a_{2k} & \diagdown & a_0 \\ b_{2k} & \diagdown & b_0 \end{smallmatrix} \in \Gamma$, or
- Γ is tractable.

Proof. Consider a functional clone $\text{Pol}(\Gamma)$ and an algebra $(A, \text{Pol}(\Gamma))$. Recall that the necessary local conditions are equivalent to requiring a conservative algebra to have only tractable 2-element subalgebras. It is obvious from the proof of Lemma 3.3 that a conservative algebra F with domain set $\{a, b\}$ is NP-hard if and only if $\begin{smallmatrix} a \\ \diagdown \\ b \end{smallmatrix} \in \text{Inv}(F)$ or $\begin{smallmatrix} b \\ \diagdown \\ a \end{smallmatrix} \in \text{Inv}(F)$ or $\text{lin}_{a,b} \in \text{Inv}(F)$. Otherwise, it is tractable. Therefore, the necessary local conditions for $\text{Pol}(\Gamma)$ are equivalent that $\forall a \neq b \in A, \begin{smallmatrix} a \\ \diagdown \\ b \end{smallmatrix} \notin \Gamma$ and $\text{lin}_{a,b} \notin \Gamma$.

Suppose Γ has the last two properties, i.e. $\text{Pol}(\Gamma)$ satisfies the necessary local conditions. As is easily seen from the proof of the NP-hard case of Theorem 3.7, Γ is NP-hard only if it contains an odd number of predicates $\begin{smallmatrix} a_0 & \diagdown & a_1 \\ b_0 & \diagdown & b_1 \end{smallmatrix}, \dots, \begin{smallmatrix} a_{2k-1} & \diagdown & a_{2k} \\ b_{2k-1} & \diagdown & b_{2k} \end{smallmatrix}, \begin{smallmatrix} a_{2k} & \diagdown & a_0 \\ b_{2k} & \diagdown & b_0 \end{smallmatrix}$. If we assume that for any $a_0 \neq b_0, \dots, a_{2k} \neq b_{2k} \in A$ this system of predicates is not contained in Γ , then Γ is tractable. ■

10. Related work and open problems

MinHom can be viewed as a problem that fits the VCSP (Valued CSP) framework by [7]. By a valued predicate of arity m over a domain D , we mean a function $p : D^m \rightarrow \mathbb{N} \cup \{\infty\}$. Informally, if Γ is a finite set of valued predicates over a finite domain D , then an instance of *VCSP*(Γ) is a set of variables together with specified subsets of variables restricted by valued predicates from Γ . Any assignment to variables can be considered a solution and the measure of this solution is the sum of the values that the valued predicates take under the assignments of the specified subsets of variables. The problem is to minimize this measure. It is widely believed that a dichotomy conjecture holds for *VCSP*(Γ), too.

Our dichotomy result for *MinHom* encourages us to consider generalizations that belong to this framework.

1. Suppose we are given a constraint language Γ and a finite set of unary functions $F \subseteq \{f : D \rightarrow \mathbb{N}\}$. Let *MinHom* _{F} (Γ) denote a minimization problem which is defined completely analogously to *MinHom*(Γ) except that we are restricted to minimizing functionals of the following form: $\sum_{i=1}^n \sum_{f \in F} w_{if} f(x_i)$. A complete classification of the complexity of this problem is an open question.

2. Suppose we have a finite valued constraint language Γ , i.e. a set of valued predicates over some finite domain set. If Γ contains all unary valued predicates, we call *VCSP*(Γ) a conservative *VCSP*. This name is motivated by the fact that in this case the polymorphisms (which is a generalization of polymorphisms for valued constraint languages [7]) of Γ must consist of conservative functions. Since there is a well-known dichotomy for conservative CSPs [5], we suspect that there is a dichotomy for conservative *VCSP*s.

3. *MinHom* has (just as CSP) a homomorphism formulation. If we restrict ourselves to relational structures given by digraphs, we arrive at the following problem which we call digraph *MinHom*: given digraphs S, H and weights $w_{ij}, i \in S, j \in H$, find a homomorphism $h : S \rightarrow H$ that minimizes the sum $\sum_{s \in S} w_{sh(s)}$. Suppose we have sets of digraphs $\mathbb{G}_1, \mathbb{G}_2$.

Then, *MinHom*($\mathbb{G}_1, \mathbb{G}_2$) denotes the digraph *MinHom* problem when the first digraph is from \mathbb{G}_1 and the second is from \mathbb{G}_2 . In this case, *MinHom*($\{H\}, All$) is always polynomially tractable and *MinHom*($All, \{H\}$) coincides with *MinHom*($\{H\}$) which is characterized in this paper. Another characterization based on digraph theory was announced during the preparation of the camera-ready version of this paper [24]. We believe that this approach could be fruitful for characterizing the complexity of *MinHom*(\mathbb{G}, \mathbb{G}): for example, is there a dichotomy for *MinHom*(\mathbb{G}, \mathbb{G})?

Acknowledgement

The author wishes to acknowledge fruitful discussions with Peter Jonsson and Andrei Bulatov.

References

- [1] Alekseev V. On the local restrictions effect on the complexity of finding the graph independence number. *Combinatorial-algebraic methods in applied mathematics*, Gorkiy University Press, 1983, pp. 3–13.
- [2] Baker K., Pixley A. Polynomial interpolation and the Chinese remainder theorem for algebraic systems. *Math. Z.*, 1975, 143, no. 2, pp. 165–174.

- [3] Bodnarcuk V.G., Kalužnin L.A., Kotov N.N., Romov B.A. Galois theory for Post algebras. *Kibernetika*, Kiev, 1969, no. 3, pp. 1–10, no. 5, pp. 1–9. (in Russian)
- [4] Brooks R. L. On colouring the nodes of a network. *Proc. Cambridge Philosophical Society*, Math. Phys. Sci., 1941, no. 37, pp. 194–197.
- [5] Bulatov A. Tractable conservative Constraint Satisfaction Problems. *Proceedings of the 18th Annual IEEE Symposium on Logic in Computer Science*, 2003, pp. 321–330.
- [6] Chudnovsky M., Robertson N., Seymour P., Thomas R. The strong perfect graph theorem. *Annals of Mathematics*, 2006, no. 164, pp. 51–229.
- [7] Cohen D., Cooper M., Jeavons P. An algebraic characterisation of complexity for valued constraints. *Proceedings of the 12th International Conference on Principles and Practice of Constraint Programming*, 2006, pp. 107–121.
- [8] Feder T., Vardi M. Y. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM Journal on Computing*, 1999, no. 28(1), pp. 57–104.
- [9] Geiger D. Closed Systems of Functions and Predicates. *Pacific Journal of Mathematics*, 1968, no. 27, pp. 95–100.
- [10] Grotshel M., Lovasz L., Schrijver A. Geometric algorithms and combinatorial optimization. Springer-Verlag, Berlin Heidelberg New York, 1988.
- [11] Grotshel M., Lovasz L., Schrijver A. Relaxations of vertex packing. *Journal of Combinatorial Theory*, 1986, Series B, no. 40(3), pp. 330–343.
- [12] Gupta A., Hell P., Karimi M., Rafiey A. Minimum cost homomorphisms to reflexive digraphs. *LATIN*, 2008.
- [13] Gutin G., Hell P., Rafiey A., Yeo A. A dichotomy for minimum cost graph homomorphisms. *European Journal of Combinatorics*, 2008, Volume 29, Issue 4, pp. 900–911.
- [14] Gutin G., Hell P., Rafiey A., Yeo A. Minimum cost and list homomorphisms to semicomplete digraphs. *Discrete Appl. Math.*, 2006, Volume 154, pp. 890–897.
- [15] Gutin G., Rafiey A., Yeo A., Tso M. Level of repair analysis and minimum cost homomorphisms of graphs. *Discrete Applied Mathematics*, no. 154(6), pp. 881–889.
- [16] Gutin G., Rafiey A., Yeo A. Minimum Cost Homomorphism Dichotomy for Oriented Cycles. *Proceedings of AAIM'08*, Lecture Notes in Computer Science, 2008, 5034, pp. 224–234.
- [17] Jeavons P. On the Algebraic Structure of Combinatorial Problems. *Theoretical Computer Science*, 1998, no. 200, 1–2, pp. 185–204.
- [18] Jégou P. Decomposition of domains based on the micro-structure of finite constraint satisfaction problems. *Proceedings of the 11th National Conference on Artificial Intelligence*, 1993, pp. 731–736.
- [19] Jonsson P. Boolean constraint satisfaction: complexity results for optimization problems with arbitrary weights. *Theoretical Computer Science*, 2000, no. 244, 1–2, pp. 189–203.
- [20] Khachiyan L. G. Polynomial algorithm in linear programming. *U.S.S.R. Comput. Math. and Math. Phys.*, 1980, no. 20, pp. 53–72.
- [21] Lovasz L. Three short proofs in graph theory. *Journal of Combinatorial Theory*, 1975, Series B, no. 19, pp. 269–271.
- [22] Marchenkov S.S. Closed classes of boolean functions. Nauka, Fizmatlit, Moscow, 2000, 126 pp. (in Russian).
- [23] Post E. The two-valued iterative systems of mathematical logic. *Annals of Mathematical Studies*, Princeton University Press, 1941, no. 5.
- [24] Rafiey A., Hell P. Duality for Min-Max Orderings and Dichotomy for Min Cost Homomorphisms. <http://arxiv.org/abs/0907.3016v1>
- [25] Schaefer T.J. The complexity of satisfiability problems. *Proc 10th ACM Symposium on Theory of Computing (STOC)*, 1978, pp. 216–226.